

UNIVERSALITY FOR PRODUCTS OF RANDOM MATRICES I: GINIBRE AND TRUNCATED UNITARY CASES

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ABSTRACT. Recently, the joint probability density functions of complex eigenvalues for products of independent complex Ginibre matrices have been explicitly derived as determinantal point processes. We express truncated series coming from the correlation kernels as multivariate integrals with singularity and investigate saddle point method for such a type of integrals. As an application, we prove that the eigenvalue correlation functions have the same scaling limits as those of the single complex Ginibre ensemble, both in the bulk and at the edge of the spectrum. We also prove that the similar results hold true for products of independent truncated unitary matrices.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. Recently, there has been a surge of interest in statistical properties of eigenvalues for products of independent random matrices with normal or more general iid entries in the literature, see [9], [17] and references therein for applications of such products. In particular, in a series of papers [2, 3, 5] Akemann and his coworkers showed that eigenvalues for products of matrices taken from complex Ginibre ensembles and truncated unitary matrices form determinantal point processes; see also [1, 27] for mixed products of Ginibre matrices, truncated unitary matrices and the inverses. These open up possibility of asymptotic analysis for local eigenvalue statistics; see [4] for a recent survey. The goal of the present paper is to investigate saddle point method for a class of multivariate integrals with singularity. As applications, in both the complex Ginibre and truncated unitary cases, we prove the pointwise convergence of one-point correlation functions to some density with a single annulus support (see [13] and [25] for the *single ring phenomenon*) and also evaluate the bulk and edge scaling limits. A forthcoming paper [40] is devoted to studying global and local eigenvalue statistics for a mixed product of Ginibre matrices, truncated unitary matrices and their inverses.

The principal subject of this paper is to study two types of products of random matrices defined as follows.

(i) Products of complex Ginibre matrices. It refers to the product of independent induced complex Ginibre ensembles $X^{(m)} = X_m X_{m-1} \cdots X_1$ where each $N \times N$ matrix X_k has the joint probability density proportional to

$$(\det(X_k^* X_k))^{a_k} e^{-\text{tr}(X_k^* X_k)}. \quad (1.1)$$

When all a_k are non-negative integers, such a product $X^{(m)}$ of square matrices can be realized as the product of rectangular matrices with normal entries where a_k represents the difference between dimensions. See [15] for the induced Ginibre ensemble and [27] for the equivalence of the two kinds of products.

(ii) Products of truncated unitary matrices. It refers to the product of independent truncated unitary matrices $X^{(m)} = X_m X_{m-1} \cdots X_1$ where each $N \times N$ matrix X_k has the joint probability density proportional to

$$(\det(X_k^* X_k))^{a_k} (\det(I_N - X_k^* X_k))^{L_k - N} 1_{\{I_N - X_k^* X_k > 0\}}, \quad (1.2)$$

where all $L_k \geq N$. In the special case where all $a_k = 0$, X_k can be treated as the top left $N \times N$ sub-block X_k of the unitary group $U(N + L_k)$ with its Haar measure (in this case if some $L_k < N$, the density of the truncated matrix X_k is different in form from (1.2) since the later is no longer applicable.), see e.g. [3] and the discussion in Appendix A therein. More generally, when all a_k are non-negative integers such a product $X^{(m)}$ of square matrices can be realized as the product of truncated rectangular sub-block from unitary matrices where a_k represents the difference between dimensions, see [27]. The product has been studied in [1, 3, 27] and the joint eigenvalue density can be found therein.

A fundamental problem in random matrix theory is to determine the limiting spectral measure of the empirical spectral distribution of a random matrix ensemble as the matrix size tends to infinity. In the non-Hermitian case the famous *circular law* for the complex Gaussian matrix was first shown by Ginibre [19]; see also Mehta [30]. For the general case when entries are iid complex variables the circular law was described by Girko [20], and was rigorously proved by Bai [6], Götze and

Tikhomirov [22], Pan and Zhou [34], and Tao and Vu [36, 37]. In particular, Tao and Vu [37] established the law with the minimum assumption that the entry has finite variance. More recently, the limit law for products of m independent complex square matrices was originally considered by Burda, Janik and Wacław [10] in the Gaussian case, and in the iid case then by Götze and Tikhomirov [23], O'Rourke and Soshnikov [32], and in the elliptic case by O'Rourke, Renfrew, Soshnikov and Vu [31]. As for the product of rectangular matrices, see [1, 11, 27] for the Gaussian case and [21, 39] for the general case.

Another fundamental problem in random matrix theory is to prove the universality of local eigenvalue statistics. In the non-Hermitian case (see [29] for a detailed review), the scaling limits in the bulk and at the edge are first computed in the complex Ginibre ensemble, see [8], [18], [16] and [30] for more details. Very recently, the same limits have been proven to be true by Fischmann etc. [15] in the induced complex Ginibre ensemble, and by Tao and Vu [38] in the iid case where entries match moments with the complex Ginibre ensemble to fourth order (Four moment theorem has been established therein). When turning to the product of random matrices, Akemann and Burda [2] calculated scaling limits in the bulk and at the edge for products of m independent complex Ginibre ensembles with each $a_k = 0$ in (1.1). In particular, a new universal limit at the origin is found, see [33] for $m = 2$ and [2] for any fixed m .

Another situation is the truncation of random unitary matrices described in [43], see [12, 28, 35] for the limiting spectral measure and [43] for local scaling limits. Products of such truncated unitary matrices have been investigated as determinantal processes, based on which the limit law for eigenvalues is obtained, see e.g. [1, 3, 27]. Also the scaling limits in the bulk and at the edge have been evaluated in [3] when each $a_k = 0$ and all the truncations L_k are equal in (1.2); the case of $L_k < N$ is also treated therein.

1.2. Main results. The joint density function of eigenvalues for the product of induced complex Ginibre ensembles $X^{(m)} = X_m X_{m-1} \cdots X_1$ with X_k distributed according to (1.1) reads off [1–3]

$$P_N(z_1, \dots, z_N) = C \prod_{l=1}^N w_m(z_l) \prod_{1 \leq j < k \leq N} |z_k - z_j|^2, \quad (1.3)$$

with respect to the Lebesgue measure on \mathbb{C}^N denoted by $\mu(dz_1) \times \cdots \times \mu(dz_N)$. Here the constant

$$C = \frac{1}{N!} \prod_{l=0}^{N-1} \prod_{k=1}^m \frac{\Gamma(a_k + 1)}{\Gamma(a_k + l + 1)},$$

and the weight function can be expressed in terms of the Meijer G-functions as

$$w_m(z) = \frac{1}{\pi} \prod_{k=1}^m \frac{1}{\Gamma(a_k + 1)} G_{0,m}^{m,0}(|z|^2 |a_1, \dots, a_m). \quad (1.4)$$

The weight function $w_m(z)$ can also be given by the high-dimensional integral

$$w_m(z) = |z|^{2a_m} \frac{2^{m-1}}{\pi} \prod_{k=1}^m \frac{1}{\Gamma(a_k + 1)} \int_{\mathbb{R}_+^{m-1}} e^{-p(r)} q(r) d^{m-1}r, \quad (1.5)$$

where

$$p(r) = p(r_1, \dots, r_{m-1}) = \sum_{k=1}^{m-1} r_k^2 + \frac{|z|^2}{r_1^2 \cdots r_{m-1}^2}, \quad (1.6)$$

and

$$q(r) = q(r_1, \dots, r_{m-1}) = \prod_{k=1}^{m-1} r_k^{2(a_k - a_m) - 1}, \quad (1.7)$$

cf. [1, 5]. This integral representation proves to be suitable for asymptotic analysis as N goes to infinity.

As a determinantal point process the kernel function reads off

$$K_N(z, z') = \sqrt{w_m(z)w_m(\bar{z}')} \sum_{l=0}^{N-1} \prod_{k=1}^m \frac{\Gamma(a_k + 1)}{\Gamma(a_k + l + 1)} (z\bar{z}')^l. \quad (1.8)$$

Actually, noticing the orthogonal relations [5]

$$\int_{\mathbb{C}} z^j \bar{z}^l w_m(z) \mu(dz) = \delta_{j,l} \prod_{k=1}^m \frac{\Gamma(a_k + l + 1)}{\Gamma(a_k + 1)}, \quad (1.9)$$

and using Dyson's integral formula [16], we have

$$P_N(z_1, \dots, z_N) = \frac{1}{N!} \det (K_N(z_j, z_k))_{1 \leq j, k \leq N}$$

and further get the n -point correlation functions

$$\begin{aligned} R_{N,n}(z_1, \dots, z_n) &:= \frac{N!}{(N-n)!} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} P_N(z) \mu(dz_{n+1}) \times \cdots \times \mu(dz_N) \\ &= \det (K_N(z_j, z_k))_{1 \leq j, k \leq n}. \end{aligned} \quad (1.10)$$

One of our main results concerns local universality for products of independent Ginibre matrices with fixed parameters, see Theorem 3.6 in Section 3 for the point-wise convergence of one-point correlation functions. In the case where all $a_k = 0$ those have been investigated by Akemann and Burda [2].

Theorem 1.1. *For fixed $a_1, \dots, a_m > -1$, let $\rho = \frac{1}{\sqrt{m}} |u|^{-(m-1)/m}$ and introduce rescaling variables*

$$z_k = u + \frac{v_k}{\rho \sqrt{N}}, \quad k = 1, \dots, n.$$

Then for $0 < |u| < 1$,

$$\begin{aligned} \lim_{N \rightarrow \infty} N^{n(m-1)} \rho^{-2n} R_{N,n}(N^{m/2} z_1, \dots, N^{m/2} z_n) \\ = \det \left(\frac{1}{\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \right)_{1 \leq k, j \leq n} \end{aligned} \quad (1.11)$$

while for $u = e^{i\phi}$ with $0 \leq \phi < 2\pi$,

$$\begin{aligned} \lim_{N \rightarrow \infty} N^{n(m-1)} \rho^{-2n} R_{N,n}(N^{m/2} z_1, \dots, N^{m/2} z_n) \\ = \det \left(\frac{1}{2\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \operatorname{erfc} \left(\frac{e^{i\phi} \bar{v}_j + e^{-i\phi} v_k}{\sqrt{2}} \right) \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (1.12)$$

Moreover, both hold true uniformly for v_1, \dots, v_n in any compact subset of \mathbb{C} .

The next two of our main results concern limiting eigenvalue density and local universality for products of independent Ginibre matrices with varying parameters. In order to use easily, for $r_{\text{in}} > 0$ we introduce a function

$$\chi_{\{u: r_{\text{in}} < |u| < r_{\text{out}}\}}(z) = \begin{cases} 1, & r_{\text{in}} < |z| < r_{\text{out}} \\ 1/2, & |z| = r_{\text{in}}, r_{\text{out}} \\ 0, & \text{otherwise,} \end{cases} \quad (1.13)$$

while for $r_{\text{in}} = 0$ we define

$$\chi_{\{u: |u| < r_{\text{out}}\}}(z) = \begin{cases} 1, & |z| < r_{\text{out}} \\ 1/2, & |z| = r_{\text{out}} \\ 0, & \text{otherwise.} \end{cases} \quad (1.14)$$

Note that in the present paper we are not concerned with the asymptotic behavior at zero. As a matter of fact, when $m > 1$ the situation is different from the single Ginibre ensemble, since in the former case the origin might be the hard edge of the spectrum (cf. Theorem 3.6 in Section 3 below). Besides, the scaling limits at the origin have been obtained in [2, 5]. Viewed from this perspective, the evaluation of the functions at zero in (1.13) and (1.14) is insignificant, cf. Theorem 1.2 below.

Theorem 1.2. *For the weight function $w_m(z)$ given by (1.5), suppose that $a_1 = \delta_1 N, \dots, a_m = \delta_m N$ with $\delta_1, \dots, \delta_m \geq 0$. For $z \neq 0$, let $\xi_m(z)$ be the largest real root of algebraic equation in x*

$$\prod_{k=1}^m (\delta_k - \delta_m + |z|^2 x) - |z|^2 = 0. \quad (1.15)$$

Then the limiting eigenvalue density

$$\begin{aligned} R_1(z) &:= \lim_{N \rightarrow \infty} N^{m-1} R_{N,1}(N^{m/2} z) \\ &= \frac{1}{\pi |z|^2} \frac{1}{\sum_{k=1}^m \frac{1}{\delta_k - \delta_m + |z|^2 \xi_m(z)}} \chi_{\{u: \sqrt{\delta_1 \cdots \delta_m} < |u| < \sqrt{(1+\delta_1) \cdots (1+\delta_m)}\}}(z) \end{aligned} \quad (1.16)$$

holds true for any complex $z \neq 0$.

Theorem 1.3. *With the same notation as in Theorem 1.2, let*

$$\rho = \frac{1}{|u| \sqrt{\sum_{k=1}^m \frac{1}{\delta_k - \delta_m + |u|^2 \xi_m(u)}}}, \quad (1.17)$$

and introduce rescaling variables

$$z_k = u + \frac{v_k}{\rho \sqrt{N}}, \quad k = 1, \dots, n. \quad (1.18)$$

Then the following hold true uniformly for v_1, \dots, v_n in any compact subset of \mathbb{C} .

(1) **Bulk limit.** For $\sqrt{\delta_1 \cdots \delta_m} < |u| < \sqrt{(1+\delta_1) \cdots (1+\delta_m)}$,

$$\begin{aligned} \lim_{N \rightarrow \infty} N^{n(m-1)} \rho^{-2n} R_{N,n}(N^{m/2} z_1, \dots, N^{m/2} z_n) \\ = \det \left(\frac{1}{\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (1.19)$$

(2) **Inner edge.** For $\delta_1, \dots, \delta_m > 0$ and $u = \sqrt{\delta_1 \cdots \delta_m} e^{i\phi}$ with $0 \leq \phi < 2\pi$,

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{n(m-1)} \rho^{-2n} R_{N,n}(N^{m/2} z_1, \dots, N^{m/2} z_n) \\ &= \det \left(\frac{1}{2\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \operatorname{erfc} \left(-\frac{e^{i\phi} \bar{v}_j + e^{-i\phi} v_k}{\sqrt{2}} \right) \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (1.20)$$

(3) **Outer edge.** For $u = \sqrt{(1 + \delta_1) \cdots (1 + \delta_m)} e^{i\phi}$ with $0 \leq \phi < 2\pi$,

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{n(m-1)} \rho^{-2n} R_{N,n}(N^{m/2} z_1, \dots, N^{m/2} z_n) \\ &= \det \left(\frac{1}{2\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \operatorname{erfc} \left(\frac{e^{i\phi} \bar{v}_j + e^{-i\phi} v_k}{\sqrt{2}} \right) \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (1.21)$$

Similar results hold true for products of truncated unitary matrices, see Theorem 5.6 for limiting eigenvalue density and Theorem 5.7 for local universality in Section 5.

Remark 1.4. We claim that the limiting eigenvalue density $R_1(z)$ given in Theorem 1.2 is symmetric with respect to the parameters $\delta_1, \dots, \delta_m$. Actually, for $z \neq 0$ and for given $j < m$, comparing the root $\xi_j(z)$ of the equation in x

$$\prod_{k=1}^m (\delta_k - \delta_j + |z|^2 x) - |z|^2 = 0 \quad (1.22)$$

and the root $\xi_m(z)$ of the equation

$$\prod_{k=1}^m (\delta_k - \delta_m + |z|^2 x) - |z|^2 = 0, \quad (1.23)$$

there is an obvious relation

$$\xi_j(z) = \frac{\delta_j - \delta_m}{|z|^2} + \xi_m(z). \quad (1.24)$$

If $\xi_m(z)$ is chosen as the largest real root, then $\xi_j(z)$ is the corresponding largest real root. In this case we know that the factor in the density function

$$\sum_{k=1}^m \frac{1}{\delta_k - \delta_j + |z|^2 \xi_j(z)} \quad (1.25)$$

is independent of j .

The paper is organized as follows. Section 2 is devoted to saddle point method for a class of multivariate principal value integrals. Two main theorems are presented and will be used to tackle with the large N asymptotics for correlation functions in next sections. In Section 3, the pointwise convergence of one-point correlation functions is proven and scaling limits in the bulk and at the edge are evaluated for products of complex Ginibre ensembles with fixed parameters a_k . The similar results are proven to hold true for products of complex Ginibre ensembles with varying parameters and for products of truncated unitary matrices respectively in Sections 4 and 5. Appendix A gives explicit expression of the determinant and the inverse for one class of matrices, which come up repeatedly in the present paper and also in the similar calculations of relevant problems.

2. ASYMPTOTICS FOR MULTIVARIATE INTEGRALS WITH SINGULARITY

To prove the bulk and edge universality, it is sufficient for us to analyse asymptotic behaviour of both the weight function and the finite sum in (1.8) in detail. Laplace's approximation (see, e.g. [42, p.495]) can be used to evaluate asymptotics of the weight function, but it does not work for the finite sum part. Our aim is to tackle the difficulty and develop a general method of asymptotically evaluating the N -term sum as $N \rightarrow \infty$. In fact, the finite sum appears ubiquitously as a factor of the correlation kernel in random matrix theory, especially in the determinant point processes associated with complex eigenvalues. We will first express the finite sum as a multivariate integral with singularity (see Section 3.1 below) and then do asymptotic analysis. In this section we devote ourselves to steepest descent method for such a type of multivariate integrals with singularity and prove the following two theorems. For convenience, we just consider the integrals with singularity at zero since the general case can be obtained with a translation.

The first result will play an important role in dealing with the universality problem in the bulk for complex eigenvalues of random matrices.

Theorem 2.1. *Let $(\theta; t) = (\theta_1, \dots, \theta_m; t)$ and $D = \gamma_1 \times \dots \times \gamma_{m+1} \subset T$, consider the integral*

$$I(\lambda) = \text{P.V.} \int_D e^{-\lambda p(\theta; t) - \sqrt{\lambda} f(\theta; t)} \frac{q(\theta; t)}{t} d^m \theta dt, \quad (2.1)$$

in which $p(\theta; t)$, $f(\theta; t)$ and $q(\theta; t)$ are independent of the positive parameter λ , single-valued and holomorphic in a domain $T \subset \mathbb{C}^{m+1}$. Here each γ_k is a smooth simple curve and is independent of λ . Suppose that $(\theta; t) = (0; 0)$ is an interior point of D such that the followings hold:

(i) *The first order partial derivatives*

$$\alpha := \frac{\partial p(0; 0)}{\partial t} \neq 0, \quad \eta := \nabla_\theta f(0; 0), \quad (2.2)$$

and

$$\nabla_\theta p(0; 0) := \left(\frac{\partial p(0; 0)}{\partial \theta_1}, \dots, \frac{\partial p(0; 0)}{\partial \theta_m} \right) = 0. \quad (2.3)$$

(ii) *For the Hessian matrix A of the first m variables,*

$$\det(A) = \det \left(\frac{\partial^2 p(0; 0)}{\partial \theta_k \partial \theta_j} \right)_{1 \leq k, j \leq m} \neq 0. \quad (2.4)$$

(iii) *There exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$, $I(\lambda)$ converges absolutely throughout its range except for one neighbourhood of $(0; 0)$.*

(iv) *For every given $\varepsilon > 0$ let $D_\varepsilon = \{(\theta; t) \in D \mid \max\{|\theta_1|, \dots, |\theta_m|, |t|\} < \varepsilon\}$, there exists $\lambda_1 > 0$ such that $\varrho(\varepsilon, \lambda_1) > 0$ where*

$$\begin{aligned} \varrho(\varepsilon, \lambda_1) = \\ \inf \left\{ \text{Re} \left\{ p(\theta; t) - p(0; 0) + \frac{f(\theta; t) - f(0; 0)}{\sqrt{\lambda}} \right\} : (\theta; t) \in D \setminus D_\varepsilon, \lambda \geq \lambda_1 \right\}. \end{aligned} \quad (2.5)$$

(v) $q(0; 0) \neq 0$.

Then as $\lambda \rightarrow \infty$ we have

$$I(\lambda) \sim i\pi(2\pi/\lambda)^{m/2} q(0; 0) e^{-\lambda p(0; 0) - \sqrt{\lambda} f(0; 0)} e^{\frac{1}{2} \eta A^{-1} \eta^T \frac{\epsilon(\alpha, A)}{\sqrt{\det A}}}, \quad (2.6)$$

where $\epsilon(\alpha, A)$ equals to 1 or -1 depending on the argument of α and the choice of the sign of $\sqrt{\det A}$.

Proof. We proceed in two steps.

Step 1: special case where $f(\theta; t) = 0$. First, the integral $I(\lambda)$ can be subdivided into two parts

$$I(\lambda) = \text{P.V.} \int_{D_\varepsilon} (\cdot) + \text{P.V.} \int_{D \setminus D_\varepsilon} (\cdot) := I_1(\lambda) + I_2(\lambda). \quad (2.7)$$

By Condition (iv), for any given $\varepsilon > 0$ there exists a positive number c such that $\text{Re}\{p(\theta; t)\} \geq c + \text{Re}\{p(0; 0)\}$ for any $(\theta; t) \in D \setminus D_\varepsilon$. Therefore, by Condition (iii) and Condition (iv)

$$\begin{aligned} |I_2(\lambda)| &= \left| \text{P.V.} \int e^{-(\lambda - \lambda_0)p(\theta; t)} e^{-\lambda_0 p(\theta; t)} \frac{q(\theta; t)}{t} d^m \theta dt \right| \\ &\leq K e^{-(\lambda - \lambda_0)(\text{Re}\{p(0; 0\}) + c)} \end{aligned} \quad (2.8)$$

for some constant $K > 0$.

If we set

$$\beta = \left(\frac{\partial^2 p(0; 0)}{\partial \theta_1 \partial t}, \dots, \frac{\partial^2 p(0; 0)}{\partial \theta_m \partial t} \right),$$

then Taylor expansion of $p(\theta; t)$ at $(0; 0)$ gives

$$\begin{aligned} p(\theta; t) &= p(0; 0) + \alpha t + \frac{1}{2} \theta A \theta^T + \frac{1}{2} \beta \theta^T t + \dots \\ &:= p(0; 0) + \alpha t (1 + S(\theta; t)) + \frac{1}{2} \theta (A + R) \theta^T. \end{aligned} \quad (2.9)$$

Here $R = (R_{kj}(\theta))$ is a symmetric matrix whose entries $R_{kj}(\theta)$ are power series of θ with $R_{kj}(0) = 0$, and $S(\theta; t)$ is a power series of $(\theta; t)$ with $S(0; 0) = 0$.

By Condition (ii), we can choose ε small enough such that the matrix $A + R$ is non-degenerate in the domain D_ε . Since $A + R$ is symmetric and non-degenerate, there exists a non-degenerate matrix Q such that $A + R = QQ^T$. Take change of variables $(\theta; t) \rightarrow (\theta'; s)$ with $\theta' = \theta Q$ and $s = \alpha t (1 + S(\theta; t))$, then for sufficiently small ε we have $\theta = \theta(\theta')$ and $t = t(s, \theta')$. Moreover, under the mapping if letting Ω be the image of $\{\theta \in \gamma_1 \times \dots \times \gamma_m \mid \max\{|\theta_1|, \dots, |\theta_m|\} < \varepsilon\}$ and for given θ' letting $\gamma'(\theta')$ be the image of $\{t \in \gamma_{m+1} \mid |t| < \varepsilon\}$, then

$$I_1(\lambda) = e^{-\lambda p(0; 0)} \int_{\Omega} e^{-\frac{1}{2} \lambda \theta' \theta'^T} d^m \theta' \left(\text{P.V.} \int_{\gamma'(\theta')} e^{-\lambda s} g(\theta'; s) \frac{1}{s} ds \right), \quad (2.10)$$

where

$$g(\theta'; s) = q(\theta; t) \det \left(\frac{\partial \theta'_k}{\partial \theta_j} \right)^{-1} \frac{1 + S(\theta; t)}{1 + S(\theta; t) + t \frac{\partial S(\theta; t)}{\partial t}}. \quad (2.11)$$

We emphasize that $\gamma'(\theta')$ is a smooth simple curve passing through 0.

On the other hand, let t_\pm be the two endpoints of the curve $\{t \in \gamma_{m+1} \mid |t| < \varepsilon\}$, by Condition (iv) there exists $\eta_1 > 0$ such that $\text{Re}\{p(\theta; t_\pm) - p(0; 0)\} \geq \eta_1$ for any $(\theta; t_\pm) \in D_\varepsilon$. Therefore, for given θ' if letting $s_\pm(\theta')$ be the image of t_\pm under the mapping, there exists $\eta_2 > 0$ such that $\text{Re}\{\frac{1}{2} \theta'^T \theta' + s_\pm(\theta')\} \geq \eta_2$ for any $\theta' \in \Omega$. This shows that we can choose a simple closed curve $\tilde{\gamma}(\theta') \supset \gamma'(\theta')$ such that

$$\text{Re} \left\{ \frac{1}{2} \theta'^T \theta' + s \right\} \geq \eta_2 \quad (2.12)$$

holds for any $\theta' \in \Omega$ and $s \in \tilde{\gamma}(\theta') \setminus \gamma'(\theta')$.

Like in the estimate of (2.8), we know from (2.12) that substitution of $\gamma'(\theta')$ with $\tilde{\gamma}(\theta')$ in (2.10) does not affect the leading order of $I_1(\lambda)$. Notice

$$\text{P.V.} \int_{\tilde{\gamma}(\theta')} e^{-\lambda s} g(\theta'; s) \frac{1}{s} ds = \text{P.V.} \int_{\tilde{\gamma}} e^{-\lambda s} g(\theta'; s) \frac{1}{s} ds,$$

where $\tilde{\gamma}$ is a smooth closed simple curve and is independent of θ' , we thus get

$$\begin{aligned} I_1(\lambda) &\sim e^{-\lambda p(0;0)} \int_{\Omega} e^{-\frac{1}{2}\lambda \theta'^T \theta'} d^m \theta' \text{P.V.} \int_{\tilde{\gamma}} e^{-\lambda s} g(\theta'; s) \frac{1}{s} ds \\ &\sim e^{-\lambda p(0;0)} \int_{\mathbb{R}^m} e^{-\frac{1}{2}\lambda \theta'^T \theta'} d^m \theta' \text{P.V.} \int_{\tilde{\gamma}} e^{-\lambda s} g(0; s) \frac{1}{s} ds \\ &= i\pi(2\pi/\lambda)^{m/2} q(0;0) e^{-\lambda p(0;0)} \frac{\epsilon(\alpha, A)}{\sqrt{\det A}} \end{aligned} \quad (2.13)$$

where we have used (2.11) and the principle value integral

$$\text{P.V.} \int_{\tilde{\gamma}} e^{-\lambda s} g(0; s) \frac{1}{s} ds = \pm i\pi g(0;0)$$

by the Sokhotskiy-Plemelj formula [26, Sect.14.1]. Here the choice of \pm depends on the orientation of integration path. The sign of $\epsilon(\alpha, A)$ relies on both the orientation of integration path and the sign of $\sqrt{\det A}$.

Combination of (2.8) and (2.13) completes the proof of Theorem 2.1 in this case.

Step 2: general case.

With (2.9) in mind the Taylor expansion of $f(\theta; t)$ at $(0;0)$ shows

$$f(\theta; t) - f(0;0) = \eta \theta^T + \frac{1}{2} \theta \tilde{R}(\theta) \theta^T + t \tilde{S}(\theta; t), \quad (2.14)$$

where $\tilde{R}(\theta)$ is a symmetric matrix whose entries are power series of θ , and $\tilde{S}(\theta; t)$ is a power series of $(\theta; t)$.

Notice Conditions (ii), (iii) and (iv), following the approach in the special case and rescaling θ and t respectively by $\frac{1}{\sqrt{\lambda}}$ and $\frac{1}{\lambda}$ near zero, the desired result could be easily derived. \square

The second result will be used to tackle the universality problem at the edge for complex eigenvalues of random matrices.

Theorem 2.2. *Let $t = (t_1, \dots, t_m, t_{m+1})$ and $D = \gamma_1 \times \dots \times \gamma_{m+1} \subset T$, consider*

$$I(\lambda) = \text{P.V.} \int_D e^{-\lambda p(t) - \sqrt{\lambda} f(t)} \frac{q(t)}{t_{m+1}} d^{m+1} t, \quad (2.15)$$

in which $p(t)$, $f(t)$ and $q(t)$ are single-valued and holomorphic in a domain $T \subset \mathbb{C}^{m+1}$. Write $\eta := (\tilde{\eta}, \eta_{m+1}) = \nabla f(0)$ where $\tilde{\eta}$ is an m -dimensional vector. Suppose that $t = 0$ is an interior point of D such that the followings hold:

(i) *The gradient of p vanishes at 0, i.e., $\nabla p(0) = 0$, or equivalently*

$$\frac{\partial p(0)}{\partial t_1} = \dots = \frac{\partial p(0)}{\partial t_m} = \frac{\partial p(0)}{\partial t_{m+1}} = 0.$$

(ii) *For the Hessian matrix A of the first m variables,*

$$\det(A) = \det \left(\frac{\partial^2 p(0)}{\partial t_k \partial t_j} \right)_{1 \leq k, j \leq m} \neq 0, \quad (2.16)$$

and the other second-order partial derivatives read off

$$\beta = \left(\frac{\partial^2 p(0)}{\partial t_1 \partial t_{m+1}}, \dots, \frac{\partial^2 p(0)}{\partial t_m \partial t_{m+1}} \right), \quad \alpha = \frac{\partial^2 p(0)}{\partial t_{m+1}^2}.$$

- (iii) There exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$, $I(\lambda)$ converges absolutely throughout its range except for one neighbourhood of 0.
- (iv) For every given $\varepsilon > 0$ let $D_\varepsilon = \{t \in D \mid \max\{|t_1|, \dots, |t_{m+1}|\} < \varepsilon\}$, there exists $\lambda_1 > 0$ such that $\varrho(\varepsilon, \lambda_1) > 0$ where

$$\varrho(\varepsilon, \lambda_1) = \inf \left\{ \operatorname{Re} \left\{ p(t) - p(0) + \frac{f(t) - f(0)}{\sqrt{\lambda}} \right\} : t \in D \setminus D_\varepsilon, \lambda \geq \lambda_1 \right\}. \quad (2.17)$$

- (v) $q(0) \neq 0$.

Then

$$\begin{aligned} I(\lambda) &\sim (2\pi/\lambda)^{m/2} q(0) e^{-\lambda p(0) - \sqrt{\lambda} f(0)} e^{\frac{1}{2} \tilde{\eta} A^{-1} \tilde{\eta}^T} \\ &\quad \times \frac{i\pi \epsilon(A)}{\sqrt{\det A}} \operatorname{erf} \left(\frac{i\eta_{m+1} - i\beta A^{-1} \tilde{\eta}^T}{\sqrt{2\alpha - 2\beta A^{-1} \beta^T}} \right) \end{aligned} \quad (2.18)$$

where $\operatorname{erf}(z)$ denotes the error function, and $\epsilon(A)$ equals to 1 or -1 depending on the choice of the sign of $\sqrt{\det(A)}$.

Proof. As in the proof of Theorem 2.1, the integral can be split into

$$I(\lambda) = \text{P.V.} \int_{D_\varepsilon} (\cdot) + \text{P.V.} \int_{D \setminus D_\varepsilon} (\cdot) := I_1(\lambda) + I_2(\lambda). \quad (2.19)$$

By Condition (iv), for sufficiently small $\varepsilon > 0$ and sufficiently large λ , there exists a positive number c independent of λ such that

$$\operatorname{Re} \left\{ p(t) - p(0) + \frac{f(t) - f(0)}{\sqrt{\lambda}} \right\} \geq c$$

for any $t \in D \setminus D_\varepsilon$. Therefore, by Condition (iii)

$$|I_2(\lambda)| \leq K e^{-(\lambda - \lambda_0)(\operatorname{Re}\{p(0)\} + c) - (\sqrt{\lambda} - \sqrt{\lambda_0}) \operatorname{Re}\{f(0)\}} \quad (2.20)$$

for some constant $K > 0$.

Next, we turn to the integral $I_1(\lambda)$. Write

$$B = \begin{pmatrix} A & \beta^T \\ \beta & \alpha \end{pmatrix},$$

then we know from Conditions (i), (ii) and (iv) that $\operatorname{Re}\{B\} > 0$. Furthermore, the Taylor expansion of $p(t)$ at $t = 0$ reads off

$$p(t) = p(0) + \frac{1}{2} t(B + R(t))t^T$$

where $R(t)$ is a symmetric matrix whose entries are power series of t with $R(0) = 0$.

As in the proof of Theorem 2.1, rescaling the variables t by $\frac{1}{\sqrt{\lambda}}$ we have the leading asymptotic form

$$\begin{aligned} I_1(\lambda) &\sim q(0)e^{-\lambda p(0)-\sqrt{\lambda}f(0)}\lambda^{-m/2} \text{P.V.} \int_{\mathbb{R}^{m+1}} e^{-\frac{1}{2}tBt^T} e^{-t\eta^T} \frac{1}{t_{m+1}} d^{m+1}t \\ &= q(0)e^{-\lambda p(0)-\sqrt{\lambda}f(0)} e^{\frac{1}{2}\tilde{\eta}A^{-1}\tilde{\eta}^T} \left(\frac{2\pi}{\lambda}\right)^{\frac{m}{2}} \frac{i\pi}{\sqrt{\det A}} \text{erf}\left(\frac{i\eta_{m+1}-i\beta A^{-1}\tilde{\eta}^T}{\sqrt{2\alpha-2\beta A^{-1}\beta^T}}\right). \end{aligned} \quad (2.21)$$

In the last equality Lemma 2.3 below has been used.

Combination of (2.20) and (2.21) completes the proof of Theorem 2.2. \square

Lemma 2.3. *Let A be an $m \times m$ complex symmetric matrix, β be an m -dimensional complex vector and $\alpha \in \mathbb{C}$. Write $t = (t_1, \dots, t_{m+1})$ and $\eta := (\tilde{\eta}, \eta_{m+1}) \in \mathbb{C}^{m+1}$ where $\tilde{\eta}$ is an m -dimensional vector. If the real part of the block matrix*

$$B = \begin{pmatrix} A & \beta^T \\ \beta & \alpha \end{pmatrix}$$

is positive definite, then

$$\begin{aligned} \text{P.V.} \int_{\mathbb{R}^{m+1}} e^{-\frac{1}{2}tBt^T} e^{-t\eta^T} \frac{1}{t_{m+1}} d^{m+1}t \\ = (2\pi)^{m/2} e^{\frac{1}{2}\tilde{\eta}A^{-1}\tilde{\eta}^T} \frac{i\pi}{\sqrt{\det A}} \text{erf}\left(\frac{i\eta_{m+1}-i\beta A^{-1}\tilde{\eta}^T}{\sqrt{2\alpha-2\beta A^{-1}\beta^T}}\right). \end{aligned} \quad (2.22)$$

Proof. First, we get $\text{Re}\{A\} > 0$ from $\text{Re}\{B\} > 0$. Since $\text{Im}\{A\}$ is symmetric by the assumption, we further have $\det A \neq 0$. Noticing the matrix decomposition

$$B = \begin{pmatrix} I & 0 \\ \beta A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \alpha - \beta A^{-1}\beta^T \end{pmatrix} \begin{pmatrix} I & A^{-1}\beta^T \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} tBt^T + 2t\eta^T &= (\tilde{t} + t_{m+1}\beta A^{-1} + \tilde{\eta}A^{-1})A(\tilde{t} + t_{m+1}\beta A^{-1} + \tilde{\eta}A^{-1})^T + \\ &\quad (\alpha - \beta A^{-1}\beta^T)t_{m+1}^2 + 2t_{m+1}(\eta_{m+1} - \beta A^{-1}\tilde{\eta}^T) - \tilde{\eta}A^{-1}\tilde{\eta}^T. \end{aligned}$$

Therefore, the Cauchy principal value integral

$$\begin{aligned} \text{P.V.} \int_{\mathbb{R}^{m+1}} e^{-\frac{1}{2}tBt^T} e^{-t\eta^T} \frac{1}{t_{m+1}} d^{m+1}t &= (2\pi)^{m/2} e^{\frac{1}{2}\tilde{\eta}A^{-1}\tilde{\eta}^T} \frac{1}{\sqrt{\det(A)}} \\ &\quad \times \text{P.V.} \int_{\mathbb{R}} e^{-\frac{1}{2}(\alpha - \beta A^{-1}\beta^T)t_{m+1}^2} e^{-(\eta_{m+1} - \beta A^{-1}\tilde{\eta}^T)t_{m+1}} \frac{1}{t_{m+1}} dt_{m+1} \end{aligned} \quad (2.23)$$

$$\begin{aligned} &= (2\pi)^{m/2} e^{\frac{1}{2}\tilde{\eta}A^{-1}\tilde{\eta}^T} \frac{1}{\sqrt{\det(A)}} \int_0^\infty dt_{m+1} e^{-\frac{1}{2}(\alpha - \beta A^{-1}\beta^T)t_{m+1}^2} \\ &\quad \times \left(\frac{e^{-(\eta_{m+1} - \beta A^{-1}\tilde{\eta}^T)t_{m+1}}}{t_{m+1}} - \frac{e^{(\eta_{m+1} - \beta A^{-1}\tilde{\eta}^T)t_{m+1}}}{t_{m+1}} \right) \end{aligned} \quad (2.24)$$

$$= (2\pi)^{m/2} e^{\frac{1}{2}\tilde{\eta}A^{-1}\tilde{\eta}^T} \frac{i\pi}{\sqrt{\det A}} \text{erf}\left(\frac{i\eta_{m+1}-i\beta A^{-1}\tilde{\eta}^T}{\sqrt{2\alpha-2\beta A^{-1}\beta^T}}\right). \quad (2.25)$$

The last equality follows from the series expression of both (2.24) and (2.25) after integrating term-by-term; see also the equation 3.321.1 [24, p.336]. \square

Remark 2.4. In order to use easily and widely, it is necessary to suppose that in Theorem 2.1 $f(\theta; t) = f_\lambda(\theta; t)$ does depend on the parameter λ . Also if

$$\lim_{\lambda \rightarrow \infty} \sqrt{\lambda} (f_\lambda(\theta/\sqrt{\lambda}; t/\lambda) - f_\lambda(0; 0)) = \eta \theta^T,$$

then the proof of Theorem 2.1 is also applicable which implies the same conclusion. Likewise, if

$$\lim_{\lambda \rightarrow \infty} \sqrt{\lambda} (f(t/\sqrt{\lambda}) - f(0)) = t \eta^T,$$

then Theorem 2.2 still holds true.

3. PRODUCTS OF GINIBRE MATRICES WITH FIXED PARAMETERS

3.1. Integral representations of finite sums. There are two main parts in the kernel function $K_N(z, z')$ given by (1.8). One is the weight function $w_m(z)$ and its integral representation in (1.5) will be used to do asymptotic analysis, see Lemma 3.2 below. The other is the truncated finite sum of hypergeometric series

$$T_N(z, z') = \sum_{l=0}^{N-1} \prod_{k=1}^m \frac{\Gamma(a_k + 1)}{\Gamma(a_k + l + 1)} (z \bar{z}')^l. \quad (3.1)$$

In order to get its asymptotics we need the following integral representation.

Proposition 3.1. *For a fixed nonzero complex number u , let $\rho = \rho(u) \neq 0$. Introduce rescaling variables*

$$z = u + \frac{v}{\rho \sqrt{N}}, \quad z' = u + \frac{v'}{\rho \sqrt{N}} \quad (3.2)$$

where v and v' lie in a compact set of \mathbb{C} . Then for sufficiently large N we have

$$\begin{aligned} T_N(N^{m/2}z, N^{m/2}z') &= (2\pi i)^{-m} N^{-\sum_{k=1}^m a_k} (z \bar{z}')^{-a_m} \prod_{j=1}^m \Gamma(a_j + 1) \\ &\times \int_{\mathcal{C}_1 \times \cdots \times \mathcal{C}_m} Q(t) e^{-\sqrt{N}F(t)} \frac{e^{-NP_1(t)} - e^{-NP_2(t)}}{t_1 \cdots t_m - 1} d^m t \end{aligned} \quad (3.3)$$

where

$$P_1(t) = -t_1 - \cdots - t_{m-1} - |u|^2 t_m, \quad (3.4)$$

$$P_2(t) = -t_1 - \cdots - t_{m-1} - |u|^2 t_m + \ln(t_1 \cdots t_m), \quad (3.5)$$

$$F(t) = -(u\bar{v}' + v\bar{u})t_m/\rho, \quad (3.6)$$

and

$$Q(t) = e^{\frac{v\bar{v}'}{\rho^2} t_m} \prod_{k=1}^m t_k^{-a_k}. \quad (3.7)$$

Here \mathcal{C}_k is a path first going from $-\infty$ to $r_k e^{i(-\pi+\theta_0)}$ ($0 < \theta_0 < \pi/2$) along the line parallel to the x -axis, then going anticlockwise along the circle with radius of r_k to $r_k e^{i(\pi-\theta_0)}$ and returning to $-\infty$ along the line parallel to the x -axis. Choose $r_1 = \cdots = r_{m-1} = |u|^{2/m}$ and $r_m = |u|^{-2(m-1)/m}$.

Proof. With the help of the integral of the reciprocal Gamma function [41, p.245]

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i} \int_{\mathcal{C}} t^{-a} e^t dt, \quad (3.8)$$

where \mathcal{C} is a path starting at $-\infty$, encircling the origin once in the positive direction and returning to $-\infty$, with (3.1) in mind we have

$$\begin{aligned} T_N(z, z') &= \prod_{j=1}^m \Gamma(a_j + 1) \sum_{l=0}^{N-1} (zz')^l \prod_{k=1}^m \left(\frac{1}{2\pi i} \int_{\mathcal{C}_k} t_k^{-(a_k+l+1)} e^{t_k} dt_k \right) \\ &= (2\pi i)^{-m} \prod_{j=1}^m \Gamma(a_j + 1) \int_{\mathcal{C}_1 \times \dots \times \mathcal{C}_m} \frac{1 - \left(\frac{zz'}{t_1 \dots t_m} \right)^N}{1 - \frac{zz'}{t_1 \dots t_m}} \prod_{k=1}^m t_k^{-(a_k+1)} e^{t_k} d^m t. \end{aligned} \quad (3.9)$$

Since

$$zz' = |u|^2 + \frac{u\bar{v}' + v\bar{u}}{\rho\sqrt{N}} + \frac{vv'}{\rho^2 N} \quad (3.10)$$

and further $\operatorname{Re}\{zz'\} > 0$ for sufficiently large N , simple calculation shows

$$\begin{aligned} T_N(N^{m/2}z, N^{m/2}z') &= (2\pi i)^{-m} N^{-\sum_{k=1}^m a_k} (zz')^{-a_m} \prod_{j=1}^m \Gamma(a_j + 1) \\ &\quad \times \int_{\mathcal{C}_1 \times \dots \times \mathcal{C}_m} Q(t) e^{-\sqrt{N}F(t)} \frac{e^{-NP_1(t)} - e^{-NP_2(t)}}{t_1 \dots t_m - 1} d^m t \end{aligned} \quad (3.11)$$

where we have used the change of variables $t_1 \rightarrow Nt_1, \dots, t_{m-1} \rightarrow Nt_{m-1}$ and $t_m \rightarrow Nzz't_m$. The paths can be chosen as needed. \square

3.2. Several lemmas. First, we compute asymptotic behaviour of the weight function.

Lemma 3.2. *For $u \neq 0$, let*

$$z = u + \frac{v}{\rho\sqrt{N}}, \quad (3.12)$$

then

$$\begin{aligned} w_m(N^{m/2}z) &\sim \frac{1}{\pi\sqrt{m}} N^{\sum_{k=1}^m a_k} \left(\frac{2\pi}{N} \right)^{(m-1)/2} \prod_{k=1}^m \frac{1}{\Gamma(a_k + 1)} \\ &\quad \times e^{-Nm|u|^{2/m} - \sqrt{N} \frac{u\bar{v} + v\bar{u}}{\rho} |u|^{-2(m-1)/m}} e^{-\frac{|v|^2}{\rho^2} |u|^{-2(m-1)/m}} \\ &\quad \times e^{\frac{m-1}{2m} \left(\frac{u\bar{v} + v\bar{u}}{\rho} \right)^2 |u|^{-4(m-1)/m} |u|^{-2/m}} |u|^{-\frac{m-1}{m} + \frac{2}{m} \sum_{k=1}^m a_k}. \end{aligned} \quad (3.13)$$

Moreover, it holds true uniformly for v in any compact subset of \mathbb{C} .

Proof. Note that by rescaling variables we have

$$\begin{aligned} w_m(N^{m/2}z) &= \frac{1}{\pi} |z|^{2a_m} 2^{m-1} N^{\sum_{k=1}^m a_k} \prod_{k=1}^m \frac{1}{\Gamma(a_k + 1)} \\ &\quad \times \int_{\mathbb{R}_+^{m-1}} e^{-\sqrt{N}f(r)} e^{-Np(r)} q(r) d^{m-1}r \end{aligned} \quad (3.14)$$

where

$$p(r) = p(r_1, \dots, r_{m-1}) = \frac{|u|^2}{r_1^2 \cdots r_{m-1}^2} + \sum_{k=1}^{m-1} r_k^2, \quad (3.15)$$

$$q(r) = q(r_1, \dots, r_{m-1}) = \prod_{k=1}^{m-1} r_k^{2(a_k - a_m) - 1} e^{-\frac{v\bar{v}}{\rho^2} \frac{1}{r_1^2 \cdots r_{m-1}^2}}, \quad (3.16)$$

and

$$f(r) = f(r_1, \dots, r_{m-1}) = \frac{u\bar{v} + v\bar{u}}{\rho} \frac{1}{r_1^2 \cdots r_{m-1}^2}. \quad (3.17)$$

One easily verifies that the point $s = (|u|^{1/m}, \dots, |u|^{1/m})$ is the unique saddle point of $p(r)$. Let

$$A = \left(\frac{\partial^2 p(s)}{\partial r_k \partial r_j} \right)_{1 \leq k, j \leq m-1},$$

where

$$\frac{\partial^2 p(s)}{\partial r_k \partial r_j} = \begin{cases} 4, & \text{if } k \neq j, \\ 8, & \text{if } k = j, \end{cases}$$

and

$$\eta := \nabla f(s) = -2 \frac{u\bar{v} + v\bar{u}}{\rho} |u|^{-2(m-1)/m} |u|^{-1/m} (1, \dots, 1).$$

We see that $p(r) + \frac{1}{\sqrt{N}} f(r)$ has a positive lower bound on the region of integration except for a neighbour of s for sufficiently large N . As in the proof of Theorem (2.1) (see Condition (iv) therein), the integral except for the part in that neighbour of s is exponentially decaying. Furthermore, Laplace's approximation [42, p.495] can be used to give

$$\begin{aligned} w_m(N^{m/2} z) &\sim \frac{|u|^{2a_m}}{\pi \sqrt{m}} N^{\sum_{k=1}^m a_k} \prod_{k=1}^m \frac{1}{\Gamma(a_k + 1)} \left(\frac{2\pi}{N} \right)^{(m-1)/2} \\ &\times e^{-Np(s)} e^{-\sqrt{N}f(s)} e^{\frac{1}{2}\eta A^{-1}\eta^T} q(s) \end{aligned} \quad (3.18)$$

from which the desired result follows. Here please refer to Appendix A for the inverse A^{-1} .

Obviously, the asymptotics holds uniformly for v in any compact subset of \mathbb{C} . \square

We remark that the asymptotics of the weight function has been computed in [2, Appendix B] when $v = 0$. Our computation is very similar to that. Actually, this special case has been studied by Barnes [7], and the readers can also find it in the Fields' work [14].

Lemma 3.3. *With the same notation as in Proposition 3.1, for sufficiently large N there exists $c_0 > 0$ such that*

$$\begin{aligned} T_N(N^{m/2} z, N^{m/2} z') &= (2\pi i)^{-m} N^{-\sum_{k=1}^m a_k} (zz')^{-a_m} \prod_{j=1}^m \Gamma(a_j + 1) \\ &\times \left(e^{Nm|u|^{2/m} - c_0 N} e^{\sqrt{N} \frac{u\bar{v}' + v\bar{u}}{\rho}} |u|^{-2(m-1)/m} O(1) \right. \\ &\quad \left. + \int_{\gamma_1 \times \cdots \times \gamma_m} q(t) e^{-\sqrt{N}f(t)} \frac{e^{-Np_1(t)} - e^{-Np_2(t)}}{t_m - 1} d^m t \right) \end{aligned} \quad (3.19)$$

where

$$p_1(t) = -t_1 - \cdots - t_{m-1} - \frac{|u|^2 t_m}{t_1 \cdots t_{m-1}}, \quad (3.20)$$

$$p_2(t) = -t_1 - \cdots - t_{m-1} - \frac{|u|^2 t_m}{t_1 \cdots t_{m-1}} + \ln t_m, \quad (3.21)$$

$$f(t) = -\frac{u\bar{v}' + v\bar{u}}{\rho} \frac{t_m}{t_1 \cdots t_{m-1}}, \quad (3.22)$$

and

$$q(t) = e^{\frac{v\bar{v}'}{\rho^2} \frac{t_m}{t_1 \cdots t_{m-1}}} t_m^{-a_m} \prod_{k=1}^{m-1} t_k^{a_m - a_k - 1}. \quad (3.23)$$

Here $\gamma_k = \{t_k : |t_k| = |u|^{2/m}, |t_k - |u|^{2/m}| < \varepsilon_0\}$ ($k = 1, \dots, m-1$) and $\gamma_m = \{t_m : |t_m| = 1, |t_m - 1| < \varepsilon_0\}$ for some $\varepsilon_0 > 0$. Moreover, both $\operatorname{Re}\{p_1(t)\}$ and $\operatorname{Re}\{p_2(t)\}$ attain their unique minimum over $\gamma_1 \times \cdots \times \gamma_m$ at $\tilde{s} = (|u|^{2/m}, \dots, |u|^{2/m}, 1)$.

Proof. Let $D = \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$ and $t_0 = (|u|^{2/m}, \dots, |u|^{2/m}, |u|^{-2+2/m})$. Note that for all $t \in D$ and $t \neq t_0$ the following relations hold

$$\operatorname{Re}\{P_1(t)\} > \operatorname{Re}\{P_1(t_0)\} \text{ and } \operatorname{Re}\{P_2(t)\} > \operatorname{Re}\{P_2(t_0)\} \text{ for } t \neq t_0. \quad (3.24)$$

Moreover, by choosing a small $\varepsilon_0 > 0$, setting

$$D_{\varepsilon_0} = \left\{ t \in D : |t_1 \cdots t_m - 1| < \varepsilon_0, |t_k - |u|^{2/m}| < \varepsilon_0, 1 \leq k < m-1 \right\},$$

one can verify that there exists a positive number c_0 such that for sufficiently large N and for every $t \in D \setminus D_{\varepsilon_0}$

$$\operatorname{Re} \left\{ P_1(t) - P_1(t_0) + \frac{F(t) - F(t_0)}{\sqrt{N}} \right\} \geq 2c_0$$

and

$$\operatorname{Re} \left\{ P_2(t) - P_2(t_0) + \frac{F(t) - F(t_0)}{\sqrt{N}} \right\} \geq 2c_0.$$

We also know from the fact $|t_1 \cdots t_m| \geq 1$ for all $t \in D$ that

$$\left| \frac{1 - e^{-N(P_2(t) - P_1(t))}}{t_1 \cdots t_m - 1} \right| = \left| \frac{1 - (t_1 \cdots t_m)^{-N}}{t_1 \cdots t_m - 1} \right| \leq N, \quad (3.25)$$

which can be controlled by $e^{-c_0 N}$. These show that the integral over $D \setminus D_{\varepsilon_0}$ is exponentially decaying as indicated in (3.19).

We next turn to the remaining integral. Noticing the choice of the contours \mathcal{C}_k and the definition of D_{ε_0} , we can always choose a small ε_0 such that the range of $t_1 \cdots t_m$ is an arc of the unit circle. By change of variables

$$(t_1, \dots, t_{m-1}, t_m) \mapsto (t_1, \dots, t_{m-1}, \frac{t_m}{t_1 \cdots t_{m-1}}),$$

the domain D_{ε_0} is changed into $\gamma_1 \times \cdots \times \gamma_m$, and the extremum property of $\operatorname{Re}\{p_1(t)\}$ and $\operatorname{Re}\{p_2(t)\}$ follows from that of $\operatorname{Re}\{P_1(t)\}$ and $\operatorname{Re}\{P_2(t)\}$ over D_{ε_0} . Then the lemma immediately follows. \square

Lemma 3.4. *For $u \neq 0$, with z, z' in (3.2) we have*

$$\begin{aligned} & (zz')^{-a_m} \text{P.V.} \int_{\gamma_1 \times \dots \times \gamma_m} q(t) e^{-\sqrt{N}f(t)} \frac{e^{-Np_1(t)}}{t_m - 1} d^m t \\ & \sim i^m \frac{\pi}{\sqrt{m}} (2\pi/N)^{(m-1)/2} |u|^{-\frac{m-1}{m} - \frac{2}{m} \sum_{k=1}^m a_k} e^{Nm|u|^{2/m} + \sqrt{N} \frac{uv' + v\bar{u}}{\rho} |u|^{-2(m-1)/m}} \\ & \quad \times e^{\frac{v\bar{v}'}{\rho^2} |u|^{-2(m-1)/m}} e^{-\frac{m-1}{2m} \left(\frac{uv' + v\bar{u}}{\rho} \right)^2 |u|^{-4(m-1)/m} |u|^{-2/m}}. \end{aligned} \quad (3.26)$$

where $p_1(t)$, $f(t)$ and $q(t)$ are given by (3.20), (3.22) and (3.23) respectively. Moreover, it holds uniformly for v, v' in any compact subset of \mathbb{C} .

Proof. Note that $\tilde{s} := (|u|^{2/m}, \dots, |u|^{2/m}, 1)$ is the unique point lying in $\gamma_1 \times \dots \times \gamma_m$ such that

$$\text{Re} \{p_1(t)\} > \text{Re} \{p_1(\tilde{s})\}, \quad t \neq \tilde{s}. \quad (3.27)$$

Moreover, for $p_1(t)$ Condition (iv) in Theorem 2.1 holds from Lemma 3.3 and others are easy to verify. By Theorem 2.1 we could do some computation and thus complete the proof.

Obviously,

$$\frac{\partial p_1(\tilde{s})}{\partial t_k} = \begin{cases} 0, & 1 \leq k < m, \\ -|u|^{2/m}, & k = m, \end{cases}$$

and the Hessian of the first $m-1$ variables reads off

$$A = \left(\frac{\partial^2 p_1(\tilde{s})}{\partial t_k \partial t_j} \right)_{1 \leq k, j \leq m-1}$$

where

$$\frac{\partial^2 p_1(\tilde{s})}{\partial t_k \partial t_j} = \begin{cases} -|u|^{-2/m}, & k \neq j, \\ -2|u|^{-2/m}, & k = j. \end{cases}$$

We also have

$$\frac{\partial f(\tilde{s})}{\partial t_k} = \frac{uv' + v\bar{u}}{\rho} |u|^{-2}.$$

Write $\eta = (\frac{\partial f(\tilde{s})}{\partial t_1}, \dots, \frac{\partial f(\tilde{s})}{\partial t_{m-1}})$, application of Theorem 2.1 gives

$$\begin{aligned} & (zz')^{-a_m} \text{P.V.} \int_{\gamma_1 \times \dots \times \gamma_m} q(t) e^{-\sqrt{N}f(t)} e^{-Np_1(t)} \frac{1}{t_m - 1} dt \\ & \sim (-1)^{m-1} |u|^{-a_m} i\pi (2\pi/N)^{(m-1)/2} q(\tilde{s}) e^{-Np_1(\tilde{s}) - \sqrt{N}f(\tilde{s})} e^{\frac{1}{2}\eta A^{-1}\eta^T} \frac{1}{\sqrt{\det A}} \end{aligned}$$

from which the lemma follows. \square

Lemma 3.5. *Let*

$$\Phi := (zz')^{-a_m} \text{P.V.} \int_{\gamma_1 \times \dots \times \gamma_m} q(t) e^{-\sqrt{N}f(t)} \frac{e^{-Np_2(t)}}{t_m - 1} d^m t \quad (3.28)$$

where $p_2(t)$, $f(t)$ and $q(t)$ are given by (3.21), (3.22) and (3.23) respectively. Then the followings hold uniformly for v, v' in any compact subset of \mathbb{C} .

(I) For $|u| \neq 0, 1$,

$$\begin{aligned} \Phi &\sim i^m \frac{\pi}{\sqrt{m}} (2\pi/N)^{(m-1)/2} |u|^{-\frac{m-1}{m} - \frac{2}{m} \sum_{k=1}^m a_k} e^{Nm|u|^{2/m} + \sqrt{N} \frac{u\bar{v}' + v\bar{u}}{\rho}} |u|^{-2(m-1)/m} \\ &\quad \times e^{\frac{v\bar{v}'}{\rho^2} |u|^{-2(m-1)/m}} e^{-\frac{m-1}{2m} \left(\frac{u\bar{v}' + v\bar{u}}{\rho} \right)^2} |u|^{-4(m-1)/m} |u|^{-2/m} \text{sign}(|u|^{2/m} - 1). \end{aligned} \quad (3.29)$$

(II) For $|u| = 1$,

$$\begin{aligned} \Phi &\sim i^m \frac{\pi}{\sqrt{m}} (2\pi/N)^{(m-1)/2} |u|^{-\frac{m-1}{m} - \frac{2}{m} \sum_{k=1}^m a_k} e^{Nm|u|^{2/m} + \sqrt{N} \frac{u\bar{v}' + v\bar{u}}{\rho}} |u|^{-2(m-1)/m} \\ &\quad \times e^{\frac{v\bar{v}'}{\rho^2} |u|^{-2(m-1)/m}} e^{-\frac{m-1}{2m} \left(\frac{u\bar{v}' + v\bar{u}}{\rho} \right)^2} |u|^{-4(m-1)/m} |u|^{-2/m} \text{erf}\left(\frac{u\bar{v}' + v\bar{u}}{\rho\sqrt{2m}}\right). \end{aligned} \quad (3.30)$$

Proof. It is easy to check that $\tilde{s} := (|u|^{2/m}, \dots, |u|^{2/m}, 1)$ is the unique point such that

$$\text{Re}\{p_2(t)\} > \text{Re}\{p_2(\tilde{s})\}, \quad t \neq \tilde{s}. \quad (3.31)$$

And moreover, for $p_2(t)$ Condition (iv) in Theorem 2.2 holds from Lemma 3.3 and others are easy to verify. Our computation is as follows.

Case (I). Simple calculation shows

$$\frac{\partial p_2(\tilde{s})}{\partial t_k} = \begin{cases} 0, & 1 \leq k < m, \\ 1 - |u|^{2/m}, & k = m, \end{cases}$$

and the Hessian of first $m-1$ variables

$$A = \left(\frac{\partial^2 p_2(\tilde{s})}{\partial t_k \partial t_j} \right)_{1 \leq k, j \leq m-1},$$

where

$$\frac{\partial^2 p_2(\tilde{s})}{\partial t_k \partial t_j} = \begin{cases} -|u|^{-2/m}, & k \neq j, \\ -2|u|^{-2/m}, & k = j. \end{cases}$$

We also have

$$\frac{\partial f(\tilde{s})}{\partial t_k} = \begin{cases} \frac{u\bar{v}' + v\bar{u}}{\rho} |u|^{-2}, & 1 \leq k < m, \\ -\frac{u\bar{v}' + v\bar{u}}{\rho} |u|^{-2(m-1)/m}, & k = m. \end{cases}$$

Let $\eta = (\frac{\partial f(\tilde{s})}{\partial t_1}, \dots, \frac{\partial f(\tilde{s})}{\partial t_{m-1}})$ and $\eta' = \frac{\partial f(\tilde{s})}{\partial t_m}$. For $|u| \neq 1$, application of Theorem 2.1 gives

$$\Phi \sim (-1)^{m-1} |u|^{-2a_m} i\pi (2\pi/N)^{(m-1)/2} q(\tilde{s}) e^{-Np_2(\tilde{s}) - \sqrt{N}f(\tilde{s})} e^{\frac{1}{2}\eta A^{-1}\eta^T} \frac{\text{sign}(|u|^{2/m} - 1)}{\sqrt{\det A}}$$

from which Statement (I) follows.

Case (II). Furthermore, we have

$$\beta = \left(\frac{\partial^2 p_2(\tilde{s})}{\partial t_1 \partial t_m}, \dots, \frac{\partial^2 p_2(\tilde{s})}{\partial t_{m-1} \partial t_m} \right) = (1, \dots, 1),$$

and

$$\alpha = \frac{\partial^2 p_2(\tilde{s})}{\partial t_m^2} = -1.$$

When $|u| = 1$, by Theorem 2.2 we get

$$\begin{aligned} \Phi &\sim (-1)^{m-1} |u|^{-2a_m} (2\pi/N)^{(m-1)/2} q(\bar{s}) e^{-Np(\bar{s}) - \sqrt{N}f(\bar{s})} e^{\frac{1}{2}\eta A^{-1}\eta^T} \\ &\quad \times \frac{i\pi}{\sqrt{\det A}} \operatorname{erf}\left(\frac{i\eta' - i\beta A^{-1}\eta^T}{\sqrt{2\alpha - 2\beta A^{-1}\beta^T}}\right) \end{aligned}$$

from which Statement (II) follows. \square

3.3. Limiting eigenvalue density. In this subsection we prove pointwise convergence of one-point correlation functions as N goes to infinity. In the parameter-fixed case (cf. Theorem 3.6 below), this result has been given in [2]; see e.g. [10] for the different convergence mode.

Theorem 3.6. *For fixed $a_1, \dots, a_m > -1$, the limiting eigenvalue density*

$$R_1(z) := \lim_{N \rightarrow \infty} N^{m-1} R_{N,1}(N^{m/2}z) = \frac{1}{m\pi} |z|^{\frac{2}{m}-2} \chi_{\{u: |u| < 1\}}(z) \quad (3.32)$$

holds true for any complex $z \neq 0$ where $\chi_{\{u: |u| < 1\}}(z)$ is defined as in (1.14).

Proof. Set $v = 0$ in Lemma 3.2, we obtain

$$\begin{aligned} w_m(N^{m/2}z) &\sim \frac{1}{\pi\sqrt{m}} (2\pi/N)^{(m-1)/2} N^{\sum_{k=1}^m a_k} \prod_{k=1}^m \frac{1}{\Gamma(a_k + 1)} \\ &\quad \times e^{-Nm|u|^{2/m}} |u|^{-\frac{m-1}{m} + \frac{2}{m} \sum_{k=1}^m a_k}. \end{aligned} \quad (3.33)$$

Set $v = v' = 0$ in Lemmas 3.3, 3.4 and 3.5, together they show

$$\begin{aligned} T_N(N^{m/2}z, N^{m/2}z) &\sim \frac{1}{\sqrt{m}} N^{-\sum_{k=1}^m a_k} (2\pi N)^{-(m-1)/2} \prod_{k=1}^m \Gamma(a_k + 1) \\ &\quad \times |u|^{-\frac{m-1}{m} - \frac{2}{m} \sum_{k=1}^m a_k} e^{Nm|u|^{2/m}} \chi_{\{z: |z| < 1\}}(u). \end{aligned} \quad (3.34)$$

Since $R_{N,1}(z) = w_m(z)T_N(z, z)$ by definition, combination of (3.33) and (3.34) completes the proof. \square

3.4. Proof of Theorem 1.1.

Proof. Let

$$\psi_N(v) = e^{-\sqrt{N} \frac{u\bar{v} - v\bar{u}}{\rho}} |u|^{-2(m-1)/m} e^{\frac{m-1}{4m} \frac{(u\bar{v})^2 - (v\bar{u})^2}{\rho^2}} |u|^{-4+2/m}, \quad (3.35)$$

define a diagonal matrix by $D = \operatorname{diag}(\psi_N(v_1), \dots, \psi_N(v_n))$.

For $0 < |u| < 1$, combining Lemmas 3.3, 3.4 and Statement (I) in Lemma 3.5, we obtain

$$\begin{aligned} T_N(N^{m/2}z_k, N^{m/2}z_j) &\sim (2\pi)^{-(m-1)/2} N^{-(m-1)/2 - \sum_{k=1}^m a_k} \prod_{j=1}^m \Gamma(a_j + 1) \\ &\quad \times \frac{1}{\sqrt{m}} |u|^{-\frac{m-1}{m} - \frac{2}{m} \sum_{k=1}^m a_k} e^{\frac{v\bar{v}'}{\rho^2} |u|^{-2(m-1)/m}} \\ &\quad \times \exp\left\{ Nm|u|^{2/m} + \sqrt{N} \frac{u\bar{v}' + v\bar{u}}{\rho} |u|^{-2(m-1)/m} \right\} \\ &\quad \times \exp\left\{ -\frac{m-1}{2m} \left(\frac{u\bar{v}' + v\bar{u}}{\rho} \right)^2 |u|^{-4(m-1)/m} |u|^{-2/m} \right\}. \end{aligned} \quad (3.36)$$

Now, combining the asymptotics of T_N and Lemmas 3.2, with the correlation kernel (1.8) in mind we find

$$\begin{aligned} \psi_N(v_k) K_N(N^{m/2} z_k, N^{m/2} z_j) \psi_N^{-1}(v_j) \\ \sim N^{-(m-1)} \frac{1}{m\pi} |u|^{-2(m-1)/m} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)}. \end{aligned} \quad (3.37)$$

Thus,

$$\begin{aligned} R_{N,n}(N^{m/2} z) &= \det \left(D(K_N(N^{m/2} z_k, N^{m/2} z_j)) D^{-1} \right) \\ &\sim N^{-n(m-1)} \rho^{2n} \det \left(\frac{1}{\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (3.38)$$

The bulk limit of (1.11) then follows.

Likewise, for $|u| = 1$, by Lemmas 3.3, 3.4 and Statement (II) in Lemma 3.5, we know

$$\begin{aligned} T_N(N^{m/2} z_k, N^{m/2} z_j) &\sim (2\pi)^{-(m-1)/2} N^{-(m-1)/2 - \sum_{k=1}^m a_k} \prod_{j=1}^m \Gamma(a_j + 1) \\ &\times \frac{1}{2\sqrt{m}} |u|^{-\frac{m-1}{m} - \frac{2}{m} \sum_{k=1}^m a_k} e^{\frac{v \bar{v}'}{\rho^2} |u|^{-2(m-1)/m}} \operatorname{erf}\left(\frac{u \bar{v}' + v \bar{u}}{\rho \sqrt{2m}}\right) \\ &\times \exp\left\{ Nm |u|^{2/m} + \sqrt{N} \frac{u \bar{v}' + v \bar{u}}{\rho} |u|^{-2(m-1)/m} \right\} \\ &\times \exp\left\{ -\frac{m-1}{2m} \left(\frac{u \bar{v}' + v \bar{u}}{\rho}\right)^2 |u|^{-4(m-1)/m} |u|^{-2/m} \right\}. \end{aligned} \quad (3.39)$$

Thus, combining the Lemma 3.2, we get

$$\begin{aligned} \psi_N(v_k) K_N(N^{m/2} z_k, N^{m/2} z_j) \psi_N^{-1}(v_j) \\ \sim N^{-(m-1)} \frac{1}{2\pi} \rho^2 e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \operatorname{erfc}\left(\frac{u \bar{v}_j + v_k \bar{u}}{\sqrt{2}}\right) \end{aligned} \quad (3.40)$$

from which the edge limit follows.

Uniform convergence on any compact subset of \mathbb{C} follows from that of the results in the involved lemmas. \square

4. PRODUCTS OF GINIBRE MATRICES WITH VARYING PARAMETERS

In this section we turn to the product of Ginibre matrices with varying parameters $a_k = \delta_k N, k = 1, \dots, m$. Without loss of generality, we assume that

$$\delta_m = \min \{\delta_k : k = 1, \dots, m\} \geq 0.$$

As in the fixed parameters case we first prove a series of lemmas and then complete the proofs of main results. For the latter use, let $\xi(u)$ be the unique positive solution of algebraic equation

$$x \prod_{k=1}^{m-1} \left((\delta_k - \delta_m) + |u|^2 x \right) - 1 = 0 \quad (4.1)$$

for any nonzero u in the complex plane.

4.1. Several lemmas.

Lemma 4.1. *Suppose that $a_k = \delta_k N, k = 1, \dots, m$. Let*

$$s_k = \sqrt{\delta_k - \delta_m + |u|^2 \xi(u)}, \quad k = 1, \dots, m-1, \quad (4.2)$$

and introduce a rescaling variable

$$z = u + \frac{v}{\rho \sqrt{N}} \quad (4.3)$$

where ρ is a real nonzero parameter. Then as $N \rightarrow \infty$ we have for $u \neq 0$

$$\begin{aligned} w_m(N^{m/2}z) &\sim \frac{\sqrt{\xi(u)}}{\pi \sqrt{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} s_k^{-2}}} \left(\frac{2\pi}{N}\right)^{(m-1)/2} \prod_{k=1}^m \frac{N^{a_k}}{\Gamma(a_k + 1)} \\ &\times \exp \left\{ -Nm |u|^2 \xi(u) - N \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - 2 \ln s_k) \right\} \\ &\times \exp \left\{ \frac{1}{2} \xi^2(u) \frac{(u\bar{v} + v\bar{u})^2}{\rho^2} \frac{1}{1 + \sum_{k=1}^{m-1} |u|^2 \xi(u) s_k^{-2}} \sum_{k=1}^{m-1} s_k^{-2} \right\} \\ &\times |z|^{2a_m} \exp \left\{ -\frac{|v|^2}{\rho^2} \xi(u) - \sqrt{N} \frac{u\bar{v} + v\bar{u}}{\rho} \xi(u) \right\}. \end{aligned} \quad (4.4)$$

Moreover, it holds true uniformly for v in any compact subset of \mathbb{C} .

Proof. The weight function (1.5) can be rewritten as

$$\begin{aligned} w_m(N^{m/2}z) &= |z|^{2a_m} \frac{2^{m-1}}{\pi} \prod_{k=1}^m \frac{N^{a_k}}{\Gamma(a_k + 1)} \\ &\times \int_{\mathbb{R}_+^{m-1}} q(r) e^{-\sqrt{N}f(r)} e^{-Np(r)} d^{m-1}r, \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} p(r) &= \sum_{k=1}^{m-1} (r_k^2 - 2(\delta_k - \delta_m) \ln r_k) + \frac{|u|^2}{\prod_{k=1}^{m-1} r_k^2}, \\ q(r) &= \exp \left\{ -\frac{|v|^2}{\rho^2} \frac{1}{\prod_{k=1}^{m-1} r_k^2} \right\} \prod_{k=1}^{m-1} \frac{1}{r_k}, \end{aligned}$$

and

$$f(r) = \frac{u\bar{v} + v\bar{u}}{\rho} \prod_{k=1}^{m-1} \frac{1}{r_k^2}.$$

We know from the first derivatives of $p(r)$

$$\frac{\partial p}{\partial r_k} = 2r_k - 2(\delta_k - \delta_m) \frac{1}{r_k} - 2 \frac{|u|^2}{r_k \prod_{j=1}^{m-1} r_j^2}$$

that $s := (s_1, \dots, s_{m-1})$ is the unique saddle point of $p(r)$ in \mathbb{R}_+^{m-1} . Also noting (4.1) we have

$$p(s) = m |u|^2 \xi(u) + \sum_{k=1}^{m-1} (\delta_k - \delta_m) \left(1 - \ln (\delta_k - \delta_m + |u|^2 \xi(u)) \right).$$

On the other hand, we can easily calculate the Hessian matrix of $p(r)$ at the point $s = (s_1, \dots, s_{m-1})$ as follows

$$A := \left(\frac{\partial^2 p(s)}{\partial r_k \partial r_j} \right)_{1 \leq k, j \leq m-1}$$

where

$$\frac{\partial^2 p(s)}{\partial r_k \partial r_j} = \begin{cases} 4|u|^2 \xi(u) \frac{1}{s_k s_j}, & k \neq j; \\ 4 + 4|u|^2 \xi(u) \frac{1}{s_k^2}, & k = j. \end{cases}$$

Therefore, A is positive definite, its inverse reads off (cf. Appendix A)

$$A^{-1} = \frac{1}{4} \frac{1}{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} s_k^{-2}} \times \begin{pmatrix} 1 + \sum_{k \neq 1} \frac{|u|^2 \xi(u)}{s_k^2} & -\frac{|u|^2 \xi(u)}{s_1 s_2} & \dots & -\frac{|u|^2 \xi(u)}{s_1 s_{m-1}} \\ -\frac{|u|^2 \xi(u)}{s_2 s_1} & 1 + \sum_{k \neq 2} \frac{|u|^2 \xi(u)}{s_k^2} & \dots & -\frac{|u|^2 \xi(u)}{s_2 s_{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{|u|^2 \xi(u)}{s_{m-1} s_1} & -\frac{|u|^2 \xi(u)}{s_{m-1} s_2} & \dots & 1 + \sum_{k \neq m-1} \frac{|u|^2 \xi(u)}{s_k^2} \end{pmatrix} \quad (4.6)$$

and the determinant of the Hessian at s is

$$\det(A) = 4^{m-1} \left(1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} s_k^{-2} \right).$$

Similarly, we can get

$$f(s) = \xi(u) \frac{u\bar{v} + v\bar{u}}{\rho}$$

and the gradient of $f(r)$ at s

$$\eta := \nabla f(s) = -2\xi(u) \frac{u\bar{v} + v\bar{u}}{\rho} \left(\frac{1}{s_1}, \dots, \frac{1}{s_{m-1}} \right).$$

Thus simple manipulation shows

$$\eta A^{-1} \eta^T = \xi^2(u) \frac{(u\bar{v} + v\bar{u})^2}{\rho^2} \frac{1}{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} s_k^{-2}} \sum_{k=1}^{m-1} s_k^{-2}.$$

We see that $p(r) + \frac{1}{\sqrt{N}} f(r)$ has a positive lower bound on the region of integration except for a neighbour of s for sufficiently large N . As in the proof of Theorem (2.1) (see Condition (iv) therein), the integral except for the part in that neighbour of s is exponentially decaying. Furthermore, Laplace's approximation [42, p.495] can be used to give

$$\begin{aligned} w_m(N^{m/2} z) &\sim |z|^{2a_m} \frac{2^{m-1}}{\pi} \prod_{k=1}^m \frac{N^{a_k}}{\Gamma(a_k + 1)} \\ &\times \left(\frac{2\pi}{N} \right)^{(m-1)/2} (\det A)^{-1/2} q(s) e^{-Np(s) - \sqrt{N}f(s)} e^{\frac{1}{2} \eta A^{-1} \eta^T} \end{aligned}$$

from which the desired result follows.

Obviously, the asymptotics holds true uniformly for v in any compact subset of \mathbb{C} . \square

Next we turn to consider the asymptotics of the truncated series given in (3.1).

Lemma 4.2. *For a fixed nonzero complex number u , let $\rho = \rho(u) \neq 0$. Suppose that $a_j = \delta_j N$ where $\delta_j \geq 0$ ($j = 1, \dots, m$) and*

$$\tilde{s}_k = \delta_k - \delta_m + |u|^2 \xi(u), \quad k = 1, \dots, m-1 \text{ while } \tilde{s}_m = 1. \quad (4.7)$$

Introduce rescaling variables

$$z = u + \frac{v}{\rho\sqrt{N}}, \quad z' = u + \frac{v'}{\rho\sqrt{N}} \quad (4.8)$$

where v, v' lie in a compact set of \mathbb{C} , then for sufficiently large N there exists $c_0 > 0$ such that

$$\begin{aligned} T_N(N^{m/2}z, N^{m/2}z') &= (2\pi i)^{-m} N^{-\sum_{k=1}^m a_k} (zz')^{-a_m} \prod_{j=1}^m \Gamma(a_j + 1) \\ &\times \left(e^{-Np_1(\tilde{s}) - c_0 N} e^{-\sqrt{N}f(\tilde{s})} O(1) \right. \\ &\quad \left. + \int_{\gamma_1 \times \dots \times \gamma_m} q(t) e^{-\sqrt{N}f(t)} \frac{e^{-Np_1(t)} - e^{-Np_2(t)}}{t_m - 1} d^m t \right) \end{aligned} \quad (4.9)$$

where

$$p_1(t) = - \sum_{k=1}^{m-1} (t_k - (\delta_k - \delta_m) \ln t_k) - \frac{|u|^2 t_m}{t_1 \cdots t_{m-1}} + \delta_m \ln t_m, \quad (4.10)$$

$$p_2(t) = - \sum_{k=1}^{m-1} (t_k - (\delta_k - \delta_m) \ln t_k) - \frac{|u|^2 t_m}{t_1 \cdots t_{m-1}} + (1 + \delta_m) \ln t_m, \quad (4.11)$$

$$f(t) = - \frac{uv' + v\bar{u}}{\rho} \frac{t_m}{t_1 \cdots t_{m-1}}, \quad (4.12)$$

and

$$q(t) = \prod_{k=1}^{m-1} t_k^{-1} e^{\frac{vv'}{\rho^2} \frac{t_m}{t_1 \cdots t_{m-1}}}. \quad (4.13)$$

Here $\gamma_k = \{t_k : |t_k| = \tilde{s}_k, |t_k - \tilde{s}_k| < \varepsilon_0\}$, $k = 1, \dots, m$ for some $\varepsilon_0 > 0$. Moreover, both $\operatorname{Re}\{p_1(t)\}$ and $\operatorname{Re}\{p_2(t)\}$ attain their unique minimum over $\gamma_1 \times \dots \times \gamma_m$ at the point $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_{m-1}, \tilde{s}_m)$.

Proof. By Proposition 3.1 we have

$$\begin{aligned} T_N(N^{m/2}z, N^{m/2}z') &= (2\pi i)^{-m} N^{-\sum_{k=1}^m a_k} (zz')^{-a_m} \prod_{j=1}^m \Gamma(a_j + 1) \\ &\times \int_{\mathcal{C}_1 \times \dots \times \mathcal{C}_m} Q(t) e^{-\sqrt{N}F(t)} \frac{e^{-NP_1(t)} - e^{-NP_2(t)}}{t_1 \cdots t_m - 1} d^m t \end{aligned} \quad (4.14)$$

where

$$P_1(t) = - \sum_{k=1}^{m-1} (t_k - \delta_k \ln t_k) - (|u|^2 t_m - \delta_m \ln t_m), \quad (4.15)$$

$$P_2(t) = - \sum_{k=1}^{m-1} (t_k - \delta_k \ln t_k) - (|u|^2 t_m - \delta_m \ln t_m) + \ln(t_1 \cdots t_m), \quad (4.16)$$

and

$$F(t) = -\frac{u\bar{v}' + v\bar{u}}{\rho}t_m, \quad Q(t) = e^{\frac{v\bar{v}'}{\rho^2}t_m}. \quad (4.17)$$

Here \mathcal{C}_k is a path first going from $-\infty$ to $r_k e^{i(-\pi+\theta_0)}$ ($0 < \theta_0 < \pi/2$) along the line parallel to the x-axis, then going anticlockwise along the circle with radius of r_k to $r_k e^{i(\pi-\theta_0)}$ and returning to $-\infty$ along the line parallel to the x-axis. Choose $r_k = \tilde{s}_k$ for any $k < m$ and $r_m = 1/(\tilde{s}_1 \cdots \tilde{s}_{m-1})$.

The subsequent procedure is the same as that in Lemma 3.3. Let $D = \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$ and $t_0 = (\tilde{s}_1, \cdots, \tilde{s}_{m-1}, 1/(\tilde{s}_1 \cdots \tilde{s}_{m-1}))$. Note that for all $t \in D$ the following relations hold true

$$\operatorname{Re} \{P_1(t)\} > \operatorname{Re} \{P_1(t_0)\} \quad \text{and} \quad \operatorname{Re} \{P_2(t)\} > \operatorname{Re} \{P_2(t_0)\} \quad \text{for } t \neq t_0. \quad (4.18)$$

Moreover, by choosing a small $\varepsilon_0 > 0$, setting

$$D_{\varepsilon_0} = \left\{ t \in D : |t_1 \cdots t_m - 1| < \varepsilon_0; |t_k - \tilde{s}_k| < \varepsilon_0, 1 \leq k < m-1 \right\},$$

one can verify that there exists a positive number c_0 such that for sufficiently large N and for every $t \in D \setminus D_{\varepsilon_0}$

$$\operatorname{Re} \left\{ P_1(t) - P_1(t_0) + \frac{F(t) - F(t_0)}{\sqrt{N}} \right\} \geq 2c_0$$

and

$$\operatorname{Re} \left\{ P_2(t) - P_2(t_0) + \frac{F(t) - F(t_0)}{\sqrt{N}} \right\} \geq 2c_0.$$

The remaining part is the same as that in Lemma 3.3 and then the lemma immediately follows. \square

Lemma 4.3. *With the same notation and assumptions as in Lemma 4.2, let*

$$\Upsilon = (zz')^{a_m} \text{P.V.} \int_{\gamma_1 \times \cdots \times \gamma_m} q(t) e^{-\sqrt{N}f(t)} e^{-Np_1(t)} \frac{1}{t_m - 1} d^m t,$$

then the following asymptotics holds true uniformly for v, v' in any compact subset of \mathbb{C} .

(I) *For $|u| \neq \sqrt{\delta_1 \cdots \delta_m}$ and $u \neq 0$,*

$$\begin{aligned} \Upsilon &\sim -i^m \pi \left(\frac{2\pi}{N} \right)^{(m-1)/2} \frac{\operatorname{sign}(\delta_m - |u|^2 \xi(u))}{\sqrt{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}} \sqrt{\xi(u)} \\ &\quad \times \exp \left\{ \xi(u) \frac{v\bar{v}'}{\rho^2} - \frac{1}{2} \xi^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}} \right\} \\ &\quad \times \exp \left\{ \sqrt{N} \xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho} + Nm |u|^2 \xi(u) + N \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - \ln \tilde{s}_k) \right\}. \end{aligned}$$

(II) For $|u| = \sqrt{\delta_1 \cdots \delta_m} \neq 0$,

$$\begin{aligned} \Upsilon &\sim i^m \pi \left(\frac{2\pi}{N} \right)^{(m-1)/2} \frac{1}{\sqrt{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}} \sqrt{\xi(u)} \\ &\times \exp \left\{ \xi(u) \frac{v\bar{v}'}{\rho^2} - \frac{1}{2} \xi^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}} \right\} \\ &\times \exp \left\{ \sqrt{N} \xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho} + Nm |u|^2 \xi(u) + N \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - \ln \tilde{s}_k) \right\} \\ &\times \operatorname{erf} \left(\frac{\sqrt{\xi(u)}}{\sqrt{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}} \frac{u\bar{v}' + v\bar{u}}{\rho \sqrt{2\delta_1 \cdots \delta_m}} \right). \end{aligned}$$

Proof. Let \tilde{s}_k be defined in (4.7), write $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_m)$. Then we know that the following inequality

$$\operatorname{Re} \{p_1(t)\} > \operatorname{Re} \{p_1(\tilde{s})\} \quad (4.19)$$

holds for all $t \in \gamma_1 \times \cdots \times \gamma_m$ except for \tilde{s} , and also

$$p_1(\tilde{s}) = -m |u|^2 \xi(u) - \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - \ln \tilde{s}_k).$$

Notice

$$\frac{\partial p_1}{\partial t_k} = \begin{cases} -1 + (\delta_k - \delta_m) \frac{1}{t_k} + \frac{|u|^2 t_m}{t_1 \cdots t_{m-1} t_k}, & k < m, \\ -\frac{|u|^2}{t_1 \cdots t_{m-1}} + \delta_m \frac{1}{t_m}, & k = m, \end{cases} \quad (4.20)$$

we have

$$\frac{\partial p_1(\tilde{s})}{\partial t_k} = 0, \quad k = 1, \dots, m-1.$$

On the other hand, for the Hessian matrix

$$A = \left(\frac{\partial^2 p_1(\tilde{s})}{\partial t_k \partial t_j} \right)_{1 \leq k, j \leq m-1},$$

where

$$\frac{\partial^2 p_1(\tilde{s})}{\partial t_k \partial t_j} = \begin{cases} -|u|^2 \xi(u) \frac{1}{\tilde{s}_k \tilde{s}_j}, & k \neq j, \\ -\frac{1}{\tilde{s}_k} - |u|^2 \xi(u) \frac{1}{\tilde{s}_k^2}, & k = j, \end{cases}$$

simple manipulation gives the determinant

$$\det(A) = (-1)^{m-1} \xi(u) \left(1 + \xi(u) |u|^2 \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k} \right) \quad (4.21)$$

and the inverse

$$A^{-1} = -\operatorname{diag}(\tilde{s}_1, \dots, \tilde{s}_{m-1}) + \frac{|u|^2 \xi(u)}{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}} 1_{(m-1) \times (m-1)}, \quad (4.22)$$

where $1_{(m-1) \times (m-1)}$ is the $(m-1) \times (m-1)$ matrix with all entries 1. Hence if letting $\eta = \left(\frac{\partial f(\tilde{s})}{\partial t_1}, \dots, \frac{\partial f(\tilde{s})}{\partial t_{m-1}} \right)$ where

$$\frac{\partial f(\tilde{s})}{\partial t_k} = \xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho} \frac{1}{\tilde{s}_k}, \quad (4.23)$$

then

$$\eta A^{-1} \eta^T = -\xi^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{1}{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\bar{s}_k}} \sum_{k=1}^{m-1} \frac{1}{\bar{s}_k}. \quad (4.24)$$

When $|u| \neq \sqrt{\delta_1 \cdots \delta_m}$ and $u \neq 0$, let $\alpha := \frac{\partial p_1(\tilde{s})}{\partial t_m} = \delta_m - |u|^2 \xi(u)$, application of Theorem 2.1 shows

$$\begin{aligned} \Upsilon &\sim (-1)^m i\pi \left(\frac{2\pi}{N} \right)^{(m-1)/2} \xi(u) e^{\xi(u) \frac{v\bar{v}'}{\rho^2}} e^{\sqrt{N} \xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho}} \\ &\quad \times e^{Nm|u|^2 \xi(u) + N \sum_{k=1}^{m-1} (\delta_k - \delta_m)(1 - \ln s_k)} e^{\frac{1}{2} \eta A^{-1} \eta^T} \frac{\text{sign}(\text{Re}\{\alpha\})}{\sqrt{\det(A)}} \end{aligned}$$

from which Statement (I) follows.

When $|u| = \sqrt{\delta_1 \cdots \delta_m} \neq 0$, we then have

$$\frac{\partial p_1(\tilde{s})}{\partial t_k} = 0, \quad k = 1, \dots, m.$$

Write

$$\beta := \left(\frac{\partial^2 p_1(\tilde{s})}{\partial t_1 \partial t_m}, \dots, \frac{\partial^2 p_1(\tilde{s})}{\partial t_{m-1} \partial t_m} \right) = |u|^2 \xi(u) \left(\frac{1}{\tilde{s}_1}, \dots, \frac{1}{\tilde{s}_{m-1}} \right),$$

and

$$\alpha' := \frac{\partial^2 p_1(\tilde{s})}{\partial t_m^2} = -\delta_m, \quad \eta_m := \frac{\partial f(\tilde{s})}{\partial t_m} = -\xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho}.$$

Following Theorem 2.2, we obtain

$$\begin{aligned} \Upsilon &\sim (-1)^{m-1} \left(\frac{2\pi}{N} \right)^{(m-1)/2} \xi(u) e^{\xi(u) \frac{v\bar{v}'}{\rho^2}} e^{\sqrt{N} \xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho}} e^{\frac{1}{2} \eta A^{-1} \eta^T} \frac{\text{sign}(\text{Re}\{\alpha'\})}{\sqrt{\det(A)}} \\ &\quad \times e^{Nm|u|^2 \xi(u) + N \sum_{k=1}^{m-1} (\delta_k - \delta_m)(1 - \ln \tilde{s}_k)} i\pi \text{erf} \left(\frac{i\eta_m - i\beta A^{-1} \eta^T}{\sqrt{2\alpha' - 2\beta A^{-1} \beta^T}} \right) \end{aligned}$$

from which Statement (II) follows.

Obviously, the asymptotics holds true uniformly for v, v' in any compact subset of \mathbb{C} . \square

Similar calculations as in the proof of Lemma 4.3 may afford us the asymptotics corresponding to the function $p_2(t)$.

Lemma 4.4. *With the same notation and assumptions as in Lemma 4.2, let*

$$\Phi = (zz')^{a_m} \text{P.V.} \int_{\gamma_1 \times \cdots \times \gamma_m} q(t) e^{-\sqrt{N}f(t)} e^{-Np_2(t)} \frac{1}{t_m - 1} d^m t,$$

then the following asymptotics holds true uniformly for v, v' in any compact subset of \mathbb{C} .

(I) For $|u| \neq \sqrt{(1+\delta_1)\cdots(1+\delta_m)}$ and $u \neq 0$,

$$\begin{aligned} \Phi &\sim -i^m \pi \left(\frac{2\pi}{N} \right)^{(m-1)/2} \frac{\text{sign}(1+\delta_m - |u|^2 \xi(u))}{\sqrt{1+|u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}} \sqrt{\xi(u)} \\ &\quad \times \exp \left\{ \xi(u) \frac{v\bar{v}'}{\rho^2} - \frac{1}{2} \xi^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}{1+|u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}} \right\} \\ &\quad \times \exp \left\{ \sqrt{N} \xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho} + Nm |u|^2 \xi(u) + N \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - \ln \tilde{s}_k) \right\}. \end{aligned}$$

(II) For $|u| = \sqrt{(1+\delta_1)\cdots(1+\delta_m)}$,

$$\begin{aligned} \Phi &\sim i^m \pi \left(\frac{2\pi}{N} \right)^{(m-1)/2} \frac{1}{\sqrt{1+|u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}} \sqrt{\xi(u)} \\ &\quad \times \exp \left\{ \xi(u) \frac{v\bar{v}'}{\rho^2} - \frac{1}{2} \xi^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}{1+|u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}} \right\} \\ &\quad \times \exp \left\{ \sqrt{N} \xi(u) \frac{u\bar{v}' + v\bar{u}}{\rho} + Nm |u|^2 \xi(u) + N \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - \ln \tilde{s}_k) \right\} \\ &\quad \times \text{erf} \left(\frac{\sqrt{\xi(u)}}{\sqrt{1+|u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}} \frac{u\bar{v}' + v\bar{u}}{\rho \sqrt{2(1+\delta_1)\cdots(1+\delta_m)}} \right). \end{aligned}$$

4.2. Proofs of Theorems 1.2 and 1.3. We are now ready to prove the limiting eigenvalue density and local universality in the parameter-varying case.

Proof of Theorems 1.2. Taking $v = 0$ in Lemma 4.1, we have

$$\begin{aligned} w_m(N^{m/2}z) &\sim \sqrt{\xi(u)} |z|^{2a_m} \left(\frac{2\pi}{N} \right)^{(m-1)/2} \prod_{k=1}^m \frac{N^{a_k}}{\Gamma(a_k + 1)} \\ &\quad \times \frac{1}{\pi \sqrt{1+|u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}} \exp \left\{ -Nm |u|^2 \xi(u) \right\} \\ &\quad \times \exp \left\{ -N \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - \ln(\delta_k - \delta_m + |u|^2 \xi(u))) \right\}. \end{aligned} \quad (4.25)$$

Taking $v, v' = 0$ in Lemmas 4.2, 4.3 and 4.4, we get the large N asymptotics of (4.9)

$$\begin{aligned} T_N(N^{m/2}z, N^{m/2}z) &\sim \pi |z|^{-2a_m} \sqrt{\xi(u)} \left(\frac{2\pi}{N} \right)^{(m-1)/2} \prod_{k=1}^m \frac{\Gamma(a_k N + 1)}{2\pi N^{a_k N}} \\ &\quad \times \exp \left\{ N \sum_{k=1}^{m-1} (\delta_k - \delta_m) (1 - \ln(\delta_k - \delta_m + |u|^2 \xi(u))) \right\} \\ &\quad \times \exp \left\{ Nm |u|^2 \xi(u) \right\} \frac{\chi_{\{z: \sqrt{\delta_1 \cdots \delta_m} < |z| < \sqrt{(1+\delta_1)\cdots(1+\delta_m)}\}}}{\sqrt{1+|u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_k}}}. \end{aligned} \quad (4.26)$$

Since $R_{N,1}(N^{m/2}z) = w_m(N^{m/2}z)T_N(N^{m/2}z, N^{m/2}z)$ and $z = u$, combination of (4.25) and (4.26) completes the proof. \square

Proof of Theorem 1.3. Let

$$\begin{aligned} \psi_N(v) = & \exp \left\{ \delta_m \sqrt{N} \frac{u\bar{v} - v\bar{u}}{2|u|^2 \rho} - \sqrt{N} \xi_m(u) \frac{u\bar{v} - v\bar{u}}{2\rho} - \delta_m \frac{(u\bar{v})^2 - (v\bar{u})^2}{4|u|^4 \rho^2} \right\} \\ & \times \exp \left\{ \frac{1}{4} \xi_m^2(u) \frac{(u\bar{v})^2 - (v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^{m-1} \frac{1}{\delta_k - \delta_m + |u|^2 \xi_m(u)}}{1 + |u|^2 \xi(u) \sum_{k=1}^{m-1} \frac{1}{\delta_k - \delta_m + |u|^2 \xi_m(u)}} \right\}, \end{aligned}$$

and let the diagonal matrix $D = \text{diag}(\psi_N(v_1), \dots, \psi_N(v_n))$.

In the bulk, combining Lemma 4.1, Lemma 4.2, Statement (I) in Lemma 4.3 and (I) in Lemma 4.4, we obtain

$$\psi_N(v_k) K_N(N^{m/2}z_k, N^{m/2}z_j) \psi_N^{-1}(v_j) \sim N^{-(m-1)} \frac{1}{\pi} \rho^2 e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)}.$$

Furthermore,

$$\begin{aligned} R_{N,n}(N^{m/2}z_1, \dots, N^{m/2}z_n) &= \det \left(D(K_N(N^{m/2}z_k, N^{m/2}z_j)) D^{-1} \right) \\ &\sim N^{-n(m-1)} \rho^{2n} \det \left(\frac{1}{\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \right)_{1 \leq k, j \leq n}. \end{aligned}$$

This completes the bulk limit of Theorem 1.3.

Likewise, combining Lemmas 4.1, 4.2, 4.3 and 4.4 we can prove the inner and outer edge cases. \square

5. PRODUCTS OF TRUNCATED UNITARY MATRICES

We take the same procedure as in Section 4 to tackle the product of independent truncated unitary matrices $X^{(m)} = X_m X_{m-1} \dots X_1$ where each X_k has the joint probability density proportional to

$$\left(\det(X_k^* X_k) \right)^{a_k} \left(\det(I_N - X_k^* X_k) \right)^{L_k - N} 1_{\{I_N - X_k^* X_k > 0\}}, \quad (5.1)$$

where all $a_k > -1$ and $L_k \geq N$. The joint eigenvalue density function of $X^{(m)}$ reads off

$$P_N(z_1, \dots, z_N) = \frac{1}{N!} \det(K_N(z_j, z_k))_{1 \leq j, k \leq N} \quad (5.2)$$

where the correlation kernel equals to

$$K_N(z, z') = \sqrt{w_m(z) w_m(\bar{z}')} T_N(z, z') \quad (5.3)$$

with all $|z_j| < 1$; as a matter of fact, the density (5.2) is applicable for all $L_k > 0$ and includes the product of truncated rectangular matrices, see [1, 3, 27]. Further, the n -point correlation functions can be expressed as

$$R_{N,n}(z_1, \dots, z_n) = \det(K_N(z_j, z_k))_{1 \leq j, k \leq n}. \quad (5.4)$$

If letting $a_k = \sigma_k N + b_k$ and $L_k = \tau_k N$ with $\sigma_1, \dots, \sigma_m \geq 0$ and $\tau_1, \dots, \tau_m > 0$, the weight function can be written as the integral (cf. Section 2.2 [3] and Eq. (99)

[27])

$$w_m(z) = \frac{2^{m-1} |z|^{2\sigma_m N + 2b_m}}{\pi} \prod_{k=1}^m \frac{\Gamma((\sigma_k + \tau_k)N + b_k + 2)}{\Gamma(\sigma_k N + b_k + 1)\Gamma(\tau_k N + 1)} \times \int_{[0,1]^{m-1}} e^{-Np(r)} q(r) d^{m-1}r, \quad (5.5)$$

where

$$p(r) = - \sum_{k=1}^{m-1} (2(\sigma_k - \sigma_m) \ln r_k + \tau_k \ln(1 - r_k^2)_+) - \tau_m \ln \left(1 - \frac{|z|^2}{\prod_{k=1}^{m-1} r_k^2} \right)_+, \quad (5.6)$$

and

$$q(r) = \prod_{k=1}^{m-1} r_k^{2(b_k - b_m) - 1}. \quad (5.7)$$

Also, the truncated series equals to

$$T_N(z, z') = \prod_{k=1}^m \frac{\Gamma(\sigma_k N + b_k + 1)}{\Gamma((\sigma_k + \tau_k)N + b_k + 2)} \sum_{l=0}^{N-1} \prod_{k=1}^m \frac{\Gamma((\sigma_k + \tau_k)N + b_k + l + 2)}{\Gamma(\sigma_k N + b_k + l + 1)} (zz')^l. \quad (5.8)$$

5.1. Integral representations. As in the Ginibre case we first rewrite the truncated series $T_N(z, z')$ as multivariate integrals.

Proposition 5.1. *For a fixed nonzero complex number u , let $\rho = \rho(u) \neq 0$. Introduce the rescaling variables*

$$z = u + \frac{v}{\rho\sqrt{N}}, \quad z' = u + \frac{v'}{\rho\sqrt{N}}, \quad (5.9)$$

where v and v' lie in a compact set of \mathbb{C} . Then

$$T_N(z, z') = N^{2m + \sum_{k=1}^m \tau_k} (zz')^{-(\sigma_m N + b_m)} (2\pi i)^{-m} \prod_{k=1}^m \frac{\Gamma(\sigma_k N + b_k + 1)}{\Gamma((\sigma_k + \tau_k)N + b_k + 2)} \times \int_{(0, \infty)^m \times \mathcal{C}_1 \times \dots \times \mathcal{C}_m} Q(t) e^{-\sqrt{N}F(t)} \frac{e^{-NP_1(t)} - e^{-NP_2(t)}}{t_{m+1} \cdots t_{2m} - 1} d^{2m}t. \quad (5.10)$$

where

$$P_1(t) = \sum_{k=1}^m (t_k - (\sigma_k - \sigma_m + \tau_k) \ln t_k) - \sum_{k=1}^{m-1} (t_{m+k} - \sigma_k \ln t_{m+k}) - (t_1 \cdots t_m) |u|^2 t_{2m} + \sigma_m \ln t_{2m}, \quad (5.11)$$

$$P_2(t) = P_1(t) + \ln(t_{m+1} \cdots t_{2m}), \quad (5.12)$$

$$Q(t) = \prod_{k=1}^m t_k^{b_k - b_m + 1} \prod_{k=1}^m t_{m+k}^{-b_k} e^{t_{2m} t_1 \cdots t_m \frac{vv'}{\rho^2}}, \quad (5.13)$$

$$F(t) = -t_{2m} t_1 \cdots t_m \frac{u\bar{v}' + v\bar{u}}{\rho}. \quad (5.14)$$

Here \mathcal{C}_k is a path first going from $-\infty$ to $r_k e^{i(-\pi + \theta_0)}$ ($0 < \theta_0 < \pi/2$) along the line parallel to the x -axis, then going anticlockwise along the circle with radius of r_k to

$r_k e^{i(\pi - \theta_0)}$ and returning to $-\infty$ along the line parallel to the x -axis. The radius r_k will be chosen properly as required.

Proof. Using integral representations of reciprocal Gamma functions (3.8) and Gamma functions, we find

$$\begin{aligned} T_N(z, z') &= \prod_{k=1}^m \frac{\Gamma(\sigma_k N + b_k + 1)}{\Gamma((\sigma_k + \tau_k)N + b_k + 2)} \sum_{l=0}^{N-1} (zz')^l \prod_{k=1}^m \int_{(0, \infty)} t_k^{(\sigma_k + \tau_k)N + b_k + l + 1} e^{-t_k} dt_k \\ &\quad \times \prod_{k=1}^m \frac{1}{2\pi i} \int_{\mathcal{C}_k} t_{m+k}^{-(\sigma_k N + b_k + l + 1)} e^{t_{m+k}} dt_{m+k} \\ &= (2\pi i)^{-m} \prod_{k=1}^m \frac{\Gamma(\sigma_k N + b_k + 1)}{\Gamma((\sigma_k + \tau_k)N + b_k + 2)} \int_{(0, \infty)^m \times \mathcal{C}_1 \times \dots \times \mathcal{C}_m} \frac{1 - \left(\frac{t_1 \dots t_m z z'}{t_{m+1} \dots t_{2m}} \right)^N}{1 - \frac{t_1 \dots t_m z z'}{t_{m+1} \dots t_{2m}}} \\ &\quad \times \prod_{k=1}^m \left(t_k^{(\sigma_k + \tau_k)N + b_k + 1} t_{m+k}^{-(\sigma_k N + b_k + 1)} e^{-t_k + t_{m+k}} \right) d^{2m} t. \end{aligned}$$

Since

$$zz' = |u|^2 + \frac{uv' + v\bar{u}}{\rho\sqrt{N}} + \frac{v\bar{v}'}{\rho^2 N},$$

taking change of variables $t_k \rightarrow Nt_k$, $k = 1, \dots, 2m-1$, and $t_{2m} \rightarrow Nt_{2m}t_1 \dots t_m z z'$ for the large N , we thus obtain the claimed result. \square

5.2. Several lemmas. Without loss of generality, we assume that

$$\sigma_m = \min \{ \sigma_k : k = 1, \dots, m \}. \quad (5.15)$$

For any $0 < |u| < 1$, we know that there exists a unique root in the interval $(|u|, 1)$ for the algebraic equation

$$\begin{aligned} x \prod_{k=1}^{m-1} \left((\tau_k + \sigma_k - \sigma_m)(x - |u|^2) + \tau_m |u|^2 \right) \\ - \prod_{k=1}^{m-1} \left((\sigma_k - \sigma_m)(x - |u|^2) + \tau_m |u|^2 \right) = 0. \end{aligned} \quad (5.16)$$

Let $\xi(u)$ be such a root.

Now the asymptotics for the weight function can be stated as follows.

Lemma 5.2. *Let*

$$s_k = \left(\frac{(\sigma_k - \sigma_m)(\xi(u) - |u|^2) + \tau_m |u|^2}{(\tau_k + \sigma_k - \sigma_m)(\xi(u) - |u|^2) + \tau_m |u|^2} \right)^{1/2}, \quad k = 1, \dots, m-1. \quad (5.17)$$

and introduce a rescaling variable

$$z = u + \frac{v}{\rho\sqrt{N}}, \quad (5.18)$$

where ρ is a real nonzero parameter depending on u . Then for $0 < |u| < 1$ as $N \rightarrow \infty$ we have

$$\begin{aligned}
w_m(z) &\sim \left(\frac{2\pi}{N}\right)^{(m-1)/2} \prod_{k=1}^m \frac{\Gamma((\sigma_k + \tau_k)N + b_k + 2)}{\Gamma(\sigma_k N + b_k + 1)\Gamma(\tau_k N + 1)} |z|^{2(\sigma_m N + b_m)} \\
&\times \frac{1}{\pi \sqrt{\left(1 + \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2} \sum_{k=1}^{m-1} \frac{(1-s_k^2)^2}{\tau_k s_k^2}\right) \prod_{k=1}^{m-1} \frac{\tau_k}{(1-s_k^2)^2}}} \prod_{k=1}^{m-1} s_k^{2(b_k - b_m) - 1} \\
&\times \exp\left\{N\tau_m \ln\left(1 - \frac{1}{\xi(u) - |u|^2} \left(\frac{u\bar{v} + v\bar{u}}{\rho\sqrt{N}} + \frac{|v|^2}{\rho^2 N}\right)\right) + N\tau_m \ln\left(1 - \frac{|u|^2}{\xi(u)}\right)\right\} \\
&\times \exp\left\{N \sum_{k=1}^{m-1} (2(\sigma_k - \sigma_m) \ln s_k + \tau_k \ln(1 - s_k^2))\right\} \\
&\times \exp\left\{\frac{1}{2} \frac{\tau_m^2 \xi^2(u)}{(\xi(u) - |u|^2)^4} \frac{(u\bar{v} + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^{m-1} \frac{(1-s_k^2)^2}{\tau_k s_k^2}}{1 + \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2} \sum_{k=1}^{m-1} \frac{(1-s_k^2)^2}{\tau_k s_k^2}}\right\}.
\end{aligned}$$

Proof. For $0 < |u| < 1$, if the variable z in the function $p(r)$ of (5.6) is replaced by u , we can verify that $s = (s_1, \dots, s_{m-1})$ is its unique saddle point and is also a minimum point. Thus it suffices for us to analyze the behavior near the neighborhood of s . Noting (5.16) and (5.17), we always choose a small neighborhood of s such that

$$1 - \frac{|u|^2}{r_1^2 \cdots r_{m-1}^2} > 0$$

Furthermore, by setting $z = u + \frac{v}{\rho\sqrt{N}}$ one gets for N large enough

$$\begin{aligned}
&\prod_{k=1}^m \frac{\Gamma(\sigma_k N + b_k + 1)\Gamma(\tau_k N + 1)}{\Gamma((\sigma_k + \tau_k)N + b_k + 2)} w_m(z) \\
&= \frac{2^{m-1} |z|^{2(\sigma_m N + b_m)}}{\pi} \int_{[0,1]^{m-1}} q(r) e^{-\sqrt{N}f_N(r)} e^{-Np(r)} d^{m-1}r,
\end{aligned} \tag{5.19}$$

where (for simplicity we use the same notation $p(r)$ as that in (5.6))

$$p(r) = - \sum_{k=1}^{m-1} (2(\sigma_k - \sigma_m) \ln r_k + \tau_k \ln(1 - r_k^2)_+) - \tau_m \ln\left(1 - \frac{|u|^2}{\prod_{k=1}^{m-1} r_k^2}\right)_+, \tag{5.20}$$

$$q(r) = \prod_{k=1}^{m-1} r_k^{2(b_k - b_m) - 1}, \tag{5.21}$$

and

$$f_N(r) = -\sqrt{N}\tau_m \ln\left(1 - \frac{1}{r_1^2 \cdots r_{m-1}^2 - |u|^2} \left(\frac{u\bar{v} + v\bar{u}}{\rho\sqrt{N}} + \frac{|v|^2}{\rho^2 N}\right)\right). \tag{5.22}$$

Next, we do some explicit calculations as follows. First, the Hessian matrix of $p(t)$ at s reads off

$$A = \left(\frac{\partial^2 p(s)}{\partial r_k \partial r_j}\right)_{1 \leq j, k \leq m-1}, \tag{5.23}$$

where

$$\frac{\partial^2 p(s)}{\partial r_k \partial r_j} = \begin{cases} 4\tau_m \frac{1}{s_k s_j} \frac{|u|^2 \xi(u)}{(\xi(u) - |u|^2)^2}, & k \neq j; \\ 4\tau_k \frac{1}{(1-s_k^2)^2} + 4\tau_m \frac{1}{s_k^2} \frac{|u|^2 \xi(u)}{(\xi(u) - |u|^2)^2}, & k = j. \end{cases} \quad (5.24)$$

It is a positive definite matrix and the inverse is equal to $A^{-1} = (A_{kj}^*)$ where for $k \neq j$

$$A_{kj}^* = -\frac{\frac{(1-s_k^2)^2}{\tau_k s_k} \frac{(1-s_j^2)^2}{\tau_j s_j} \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2}}{1 + \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2} \sum_{l=1}^{m-1} \frac{(1-s_l^2)^2}{\tau_l s_l^2}} \quad (5.25)$$

and for $k = j$

$$A_{kk}^* = \frac{(1-s_k^2)^2}{\tau_k} \frac{1 + \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2} \sum_{l \neq k} \frac{(1-s_l^2)^2}{\tau_l s_l^2}}{1 + \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2} \sum_{l=1}^{m-1} \frac{(1-s_l^2)^2}{\tau_l s_l^2}}. \quad (5.26)$$

Secondly, the determinant of the Hessian at s is equal to

$$\det(A) = 4^{m-1} \left(1 + \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2} \sum_{k=1}^{m-1} \frac{(1-s_k^2)^2}{\tau_k s_k^2} \right) \prod_{k=1}^{m-1} \frac{\tau_k}{(1-s_k^2)^2} \quad (5.27)$$

and

$$\eta := \lim_{N \rightarrow \infty} \nabla f_N(s) = -2 \frac{\tau_m \xi(u)}{(\xi(u) - |u|^2)^2} \frac{u\bar{v} + v\bar{u}}{\rho} \left(\frac{1}{s_1}, \dots, \frac{1}{s_{m-1}} \right). \quad (5.28)$$

We see that $p(r) + \frac{1}{\sqrt{N}} f_N(r)$ has a positive lower bound on the region of integration except for a neighbour of s for sufficiently large N . As in the proof of Theorem (2.1) (see Condition (iv) therein and Remark 2.4), the integral except for the part in that neighbour of s is exponentially decaying. Furthermore, Laplace's approximation [42, p.495] can be used to give

$$\begin{aligned} w_m(z) &\sim \frac{2^{m-1} |z|^{2(\sigma_m N + b_m)}}{\pi} \prod_{k=1}^m \frac{\Gamma((\sigma_k + \tau_k)N + b_k + 2)}{\Gamma(\sigma_k N + b_k + 1) \Gamma(\tau_k N + 1)} \\ &\times \left(\frac{2\pi}{N} \right)^{(m-1)/2} (\det(A))^{-1/2} q(s) e^{-Np(s) - \sqrt{N} f_N(s)} e^{\frac{1}{2} \eta A^{-1} \eta^T} \end{aligned} \quad (5.29)$$

from which the desired result follows. \square

We now turn to consider the asymptotics of the truncated series given in (5.8). For $u \neq 0$, let $\zeta(u)$ be the unique positive solution of the algebraic equation

$$\prod_{k=1}^m \frac{\sigma_k - \sigma_m + x |u|^2}{\sigma_k - \sigma_m + \tau_k + x |u|^2} - |u|^2 = 0. \quad (5.30)$$

For $1 \leq k \leq m$ and $1 \leq j \leq m-1$ set

$$\tilde{s}_k = \sigma_k - \sigma_m + \tau_k + \zeta(u) |u|^2, \quad \tilde{s}_{m+j} = \sigma_j - \sigma_m + \zeta(u) |u|^2, \quad \tilde{s}_{2m} = 1. \quad (5.31)$$

Then we have the following lemma. Its proof is just the same as that of Lemma 4.2 and thus may be omitted.

Lemma 5.3. *Suppose that u is a fixed nonzero complex number and \tilde{s}_k is defined as in (5.31). Let $\rho = \rho(u) \neq 0$ and introduce rescaling variables*

$$z = u + \frac{v}{\rho \sqrt{N}}, \quad z' = u + \frac{v'}{\rho \sqrt{N}} \quad (5.32)$$

where v and v' lie in a compact set of \mathbb{C} , then for sufficiently large N there exists $c_0 > 0$ such that

$$\begin{aligned} T_N(z, z') &= N^{2m + \sum_{k=1}^m \tau_k} (zz')^{-(\sigma_m N + b_m)} (2\pi i)^{-m} \prod_{k=1}^m \frac{\Gamma(\sigma_k N + b_k + 1)}{\Gamma(\sigma_k N + \tau_k N + b_k + 2)} \\ &\times \left(e^{-N p_1(\tilde{s}) - c_0 N} e^{-\sqrt{N} f(\tilde{s})} O(1) \right. \\ &\quad \left. + \int_{\gamma_1 \times \cdots \times \gamma_{2m}} q(t) e^{-\sqrt{N} f(t)} \frac{e^{-N p_1(t)} - e^{-N p_2(t)}}{t_{2m} - 1} d^{2m} t \right) \end{aligned} \quad (5.33)$$

where

$$\begin{aligned} p_1(t) &= \sum_{k=1}^m (t_k - (\sigma_k - \sigma_m + \tau_k) \ln t_k) - \frac{t_1 \cdots t_m}{t_{m+1} \cdots t_{2m-1}} |u|^2 t_{2m} \\ &\quad - \sum_{k=1}^{m-1} (t_{m+k} - (\sigma_k - \sigma_m) \ln t_{m+k}) + \sigma_m \ln t_{2m}, \end{aligned} \quad (5.34)$$

$$p_2(t) = p_1(t) + \ln t_{2m}, \quad (5.35)$$

$$q(t) = \left(\prod_{k=1}^m t_k^{b_k - b_m + 1} \right) \left(\prod_{k=1}^{m-1} t_{m+k}^{-(b_k - b_m + 1)} \right) t_{2m}^{-b_m} e^{\frac{t_1 \cdots t_m}{t_{m+1} \cdots t_{2m-1}} \frac{v \bar{v}'}{\rho^2} t_{2m}}, \quad (5.36)$$

and

$$f(t) = -\frac{t_1 \cdots t_m}{t_{m+1} \cdots t_{2m-1}} \frac{u \bar{v}' + v \bar{u}}{\rho} t_{2m}. \quad (5.37)$$

Here $\gamma_k = \{t_k : -\varepsilon_0 < t_k - \tilde{s}_k < \varepsilon_0\}$ and $\gamma_{m+k} = \{t_{m+k} : |t_{m+k} - \tilde{s}_{m+k}, |t_{m+k} - \tilde{s}_{m+k}| < \varepsilon_0\}$ for some $\varepsilon_0 > 0$ ($k = 1, \dots, m$). Moreover, both $\operatorname{Re}\{p_1(t)\}$ and $\operatorname{Re}\{p_2(t)\}$ attain their unique minimum over $\gamma_1 \times \cdots \times \gamma_{2m}$ at the point $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_{m-1}, \tilde{s}_{2m})$.

Lemma 5.4. *With the same notation and assumptions as in Lemma 5.3, let*

$$\Upsilon = \text{P.V.} \int_{\gamma_1 \times \cdots \times \gamma_{2m}} q(t) e^{-\sqrt{N} f(t)} e^{-N p_1(t)} \frac{1}{t_{2m} - 1} d^{2m} t, \quad (5.38)$$

then the following hold true uniformly for v, v' in any compact subset of \mathbb{C} .

(I) For $|u| \neq \sqrt{\frac{\sigma_1 \cdots \sigma_m}{(\sigma_1 + \tau_1) \cdots (\sigma_m + \tau_m)}}$,

$$\begin{aligned} \Upsilon &\sim \frac{-i^m \pi (2\pi/N)^{(2m-1)/2} \operatorname{sign}(\sigma_m - \zeta(u)|u|^2)}{\pi \sqrt{(1 - \zeta(u)|u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})) \prod_{k=1}^{2m-1} \frac{1}{\tilde{s}_k}}} \\ &\times \exp \left\{ -N \sum_{k=1}^m (\tilde{s}_k - (\sigma_k - \sigma_m + \tau_k) \ln \tilde{s}_k) + N \zeta(u) |u|^2 \right\} \\ &\times \exp \left\{ N \sum_{k=1}^{m-1} (\tilde{s}_{m+k} - (\sigma_k - \sigma_m) \ln \tilde{s}_{m+k}) - N \sigma_m \right\} \\ &\times \exp \left\{ \sqrt{N} \zeta(u) \frac{u \bar{v}' + v \bar{u}}{\rho} \right\} \prod_{k=1}^m \tilde{s}_k^{b_k - b_m + 1} \prod_{l=1}^{m-1} \tilde{s}_{m+l}^{-(b_l - b_m + 1)} e^{\zeta(u) \frac{v \bar{v}'}{\rho^2}} \\ &\times \exp \left\{ \frac{1}{2} \zeta^2(u) \frac{(u \bar{v}' + v \bar{u})^2}{\rho^2} \frac{\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}}}{1 - \zeta(u) |u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})} \right\}. \end{aligned}$$

(II) For $|u| = \sqrt{\frac{\sigma_1 \cdots \sigma_m}{(\sigma_1 + \tau_1) \cdots (\sigma_m + \tau_m)}} \neq 0$,

$$\begin{aligned} \Upsilon &\sim \frac{i^m \pi (2\pi/N)^{(2m-1)/2}}{\pi \sqrt{(1 - \zeta(u)|u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})) \prod_{k=1}^{2m-1} \frac{1}{\tilde{s}_k}}} \\ &\times \exp \left\{ -N \sum_{k=1}^m (\tilde{s}_k - (\sigma_k - \sigma_m + \tau_k) \ln \tilde{s}_k) + N \zeta(u) |u|^2 \right\} \\ &\times \exp \left\{ N \sum_{k=1}^{m-1} (\tilde{s}_{m+k} - (\sigma_k - \sigma_m) \ln \tilde{s}_{m+k}) - N \sigma_m \right\} \\ &\times \exp \left\{ \sqrt{N} \zeta(u) \frac{u \bar{v}' + v \bar{u}}{\rho} \right\} \prod_{k=1}^m \tilde{s}_k^{b_k - b_m + 1} \prod_{l=1}^{m-1} \tilde{s}_{m+l}^{-(b_l - b_m + 1)} e^{\zeta(u) \frac{v \bar{v}'}{\rho^2}} \\ &\times \exp \left\{ \frac{1}{2} \zeta^2(u) \frac{(u \bar{v}' + v \bar{u})^2}{\rho^2} \frac{\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}}}{1 - \zeta(u) |u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})} \right\} \\ &\times \operatorname{erf} \left(\frac{\sqrt{\zeta(u)}}{\sqrt{1 - \zeta(u) |u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})}} \frac{u \bar{v}' + v \bar{u}}{\rho \sqrt{2} |u|} \right). \end{aligned}$$

Proof. For \tilde{s}_k given in (5.31), write $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_{2m})$. Then we know that the following inequality

$$\operatorname{Re} \left\{ p_1(t) + \frac{1}{\sqrt{N}} f(t) \right\} > \operatorname{Re} \left\{ p_1(\tilde{s}) + \frac{1}{\sqrt{N}} f(\tilde{s}) \right\}$$

holds for all $t \in \gamma_1 \times \cdots \times \gamma_{2m}$ except for \tilde{s} .

Notice

$$\begin{aligned}\frac{\partial p_1(t)}{\partial t_k} &= 1 - (\sigma_k - \sigma_m + \tau_k) \frac{1}{t_k} - \frac{1}{t_k} \frac{t_1 \cdots t_m |u|^2 t_{2m}}{t_{m+1} \cdots t_{2m-1}}, \quad 1 \leq k \leq m; \\ \frac{\partial p_1(t)}{\partial t_{m+k}} &= -1 + (\sigma_k - \sigma_m) \frac{1}{t_k} + \frac{1}{t_k} \frac{t_1 \cdots t_m |u|^2 t_{2m}}{t_{m+1} \cdots t_{2m-1}}, \quad 1 \leq k < m; \\ \frac{\partial p_1(t)}{\partial t_{2m}} &= -\frac{t_1 \cdots t_m |u|^2}{t_{m+1} \cdots t_{2m-1}} + \sigma_m \frac{1}{t_{2m}},\end{aligned}$$

we have

$$\frac{\partial p_1(\tilde{s})}{\partial t_k} = 0, \quad k = 1, \dots, 2m-1.$$

On the other hand, for the Hessian matrix

$$A = \left(\frac{\partial^2 p_1(\tilde{s})}{\partial t_k \partial t_j} \right)_{1 \leq k, j \leq 2m-1} \quad (5.39)$$

where

$$\frac{\partial^2 p_1(\tilde{s})}{\partial t_k \partial t_j} = \begin{cases} -\frac{1}{\tilde{s}_k \tilde{s}_j} \zeta(u) |u|^2, & 1 \leq k \neq j \leq m \text{ or } m < k \neq j < 2m; \\ \frac{1}{\tilde{s}_k \tilde{s}_j} \zeta(u) |u|^2, & 1 \leq k \leq m < j < 2m; \\ \frac{1}{\tilde{s}_k} - \frac{1}{\tilde{s}_k^2} \zeta(u) |u|^2, & 1 \leq k = j \leq m; \\ -\frac{1}{\tilde{s}_k} - \frac{1}{\tilde{s}_k^2} \zeta(u) |u|^2, & m < k = j < 2m, \end{cases} \quad (5.40)$$

simple manipulation gives the determinant

$$\det(A) = (-1)^{m-1} \left(1 - \zeta(u) |u|^2 \left(\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}} \right) \right) \prod_{k=1}^{2m-1} \frac{1}{\tilde{s}_k} \quad (5.41)$$

and the inverse

$$\begin{aligned}A^{-1} &= \text{diag}(\tilde{s}_1, \dots, \tilde{s}_m, -\tilde{s}_{m+1}, \dots, -\tilde{s}_{2m-1}) \\ &\quad + \frac{\zeta(u) |u|^2}{1 - \zeta(u) |u|^2 \left(\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}} \right)} \mathbf{1}_{(2m-1) \times (2m-1)},\end{aligned} \quad (5.42)$$

where $\mathbf{1}_{(2m-1) \times (2m-1)}$ is a $(2m-1) \times (2m-1)$ matrix with all entries 1.

Write $\eta = \left(\frac{\partial f(\tilde{s})}{\partial t_1}, \dots, \frac{\partial f(\tilde{s})}{\partial t_{2m-1}} \right)$ where

$$\frac{\partial f(\tilde{s})}{\partial t_k} = \begin{cases} -\zeta(u) \frac{u\bar{v}' + v\bar{u}}{\rho} \frac{1}{\tilde{s}_k}, & k = 1, \dots, m, \\ \zeta(u) \frac{u\bar{v}' + v\bar{u}}{\rho} \frac{1}{\tilde{s}_k}, & k = m+1, \dots, 2m-1, \end{cases} \quad (5.43)$$

then

$$\eta A^{-1} \eta^T = \zeta^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}}}{1 - \zeta(u) |u|^2 \left(\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}} \right)}. \quad (5.44)$$

When $|u| \neq \sqrt{\frac{\sigma_1 \cdots \sigma_m}{(\sigma_1 + \tau_1) \cdots (\sigma_m + \tau_m)}}$, let

$$\alpha = \frac{\partial p_1(\tilde{s})}{\partial t_{2m}} = \sigma_m - \zeta(u) |u|^2, \quad (5.45)$$

application of Theorem 2.1 gives

$$\Upsilon \sim (-1)^m i\pi (2\pi/\lambda)^{(2m-1)/2} q(\tilde{s}) e^{-\lambda p_1(\tilde{s}) - \sqrt{\lambda} f(\tilde{s})} e^{\frac{1}{2}\eta A^{-1}\eta^T} \frac{\text{sign}(\alpha)}{\sqrt{\det A}} \quad (5.46)$$

from which Statement (I) follows.

If $|u| = \sqrt{\frac{\sigma_1 \cdots \sigma_m}{(\sigma_1 + \tau_1) \cdots (\sigma_m + \tau_m)}} \neq 0$, we then have

$$\frac{\partial p_1(\tilde{s})}{\partial t_k} = 0, \quad k = 1, \dots, 2m. \quad (5.47)$$

Write

$$\beta = \zeta(u) |u|^2 \left(-\frac{1}{\tilde{s}_1}, \dots, -\frac{1}{\tilde{s}_m}, \frac{1}{\tilde{s}_{m+1}}, \dots, \frac{1}{\tilde{s}_{2m-1}} \right), \quad (5.48)$$

and

$$\alpha' = \frac{\partial^2 p_1(\tilde{s})}{\partial t_{2m}^2} = -\sigma_m, \quad \eta_{2m} = \frac{\partial f(\tilde{s})}{\partial t_{2m}} = -\zeta(u) \frac{u\bar{v}' + v\bar{u}}{\rho}. \quad (5.49)$$

By Theorem 2.2, we obtain

$$\Upsilon \sim (-1)^{m-1} (2\pi/\lambda)^{m/2} q(\tilde{s}) e^{-\lambda p(\tilde{s}) - \sqrt{\lambda} f(\tilde{s})} e^{\frac{1}{2}\eta A^{-1}\eta^T} \frac{i\pi}{\sqrt{\det A}} \text{erf}\left(\frac{i\eta_{2m} - i\beta A^{-1}\eta^T}{\sqrt{2\alpha' - 2\beta A^{-1}\beta^T}}\right) \quad (5.50)$$

from which Statement (II) follows.

Obviously, the asymptotics holds true uniformly for v, v' in any compact subset of \mathbb{C} . \square

Similar calculations as in the proof of Lemma 5.4 may afford us the asymptotics corresponding to the function $p_2(t)$.

Lemma 5.5. *With the same notation and assumptions as in Lemma 5.3, let*

$$\Phi = \text{P.V.} \int_{\gamma_1 \times \cdots \times \gamma_{2m}} q(t) e^{-\sqrt{N}f(t)} e^{-Np_2(t)} \frac{1}{t_{2m} - 1} d^{2m}t, \quad (5.51)$$

then the following hold true uniformly for v, v' in any compact subset of \mathbb{C} .

$$(I) \text{ For } |u| \neq \sqrt{\frac{(1+\sigma_1) \cdots (1+\sigma_m)}{(1+\sigma_1+\tau_1) \cdots (1+\sigma_m+\tau_m)}},$$

$$\begin{aligned} \Phi \sim & \frac{-i^m \pi (2\pi/N)^{(2m-1)/2} \text{sign}(1 + \sigma_m - \zeta(u) |u|^2)}{\pi \sqrt{(1 - \zeta(u) |u|^2) \left(\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}} \right) \prod_{k=1}^{2m-1} \frac{1}{\tilde{s}_k}}} \\ & \times \exp \left\{ -N \sum_{k=1}^m \left(\tilde{s}_k - (\sigma_k - \sigma_m + \tau_k) \ln \tilde{s}_k \right) + N \zeta(u) |u|^2 \right\} \\ & \times \exp \left\{ N \sum_{k=1}^{m-1} \left(\tilde{s}_{m+k} - (\sigma_k - \sigma_m) \ln \tilde{s}_{m+k} \right) - N \sigma_m \right\} \\ & \times \exp \left\{ \sqrt{N} \zeta(u) \frac{u\bar{v}' + v\bar{u}}{\rho} \right\} \prod_{k=1}^m \tilde{s}_k^{b_k - b_m + 1} \prod_{l=1}^{m-1} \tilde{s}_{m+l}^{-(b_l - b_m + 1)} e^{\zeta(u) \frac{v\bar{v}'}{\rho^2}} \\ & \times \exp \left\{ \frac{1}{2} \zeta^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}}}{1 - \zeta(u) |u|^2 \left(\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}} \right)} \right\} \end{aligned}$$

$$\begin{aligned}
(\text{II}) \quad & \text{For } |u| = \sqrt{\frac{(1+\sigma_1)\cdots(1+\sigma_m)}{(1+\sigma_1+\tau_1)\cdots(1+\sigma_m+\tau_m)}}, \\
\Phi \sim & \frac{i^m \pi (2\pi/N)^{(2m-1)/2}}{\pi \sqrt{(1-\zeta(u)|u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{k=1}^{m-1} \frac{1}{\tilde{s}_{m+k}})) \prod_{k=1}^{2m-1} \frac{1}{\tilde{s}_k}}} \\
& \times \exp \left\{ -N \sum_{k=1}^m (\tilde{s}_k - (\sigma_k - \sigma_m + \tau_k) \ln \tilde{s}_k) + N \zeta(u) |u|^2 \right\} \\
& \times \exp \left\{ N \sum_{k=1}^{m-1} (\tilde{s}_{m+k} - (\sigma_k - \sigma_m) \ln \tilde{s}_{m+k}) - N \sigma_m \right\} \\
& \times \exp \left\{ \sqrt{N} \zeta(u) \frac{u\bar{v}' + v\bar{u}}{\rho} \right\} \prod_{k=1}^m \tilde{s}_k^{b_k - b_m + 1} \prod_{l=1}^{m-1} \tilde{s}_{m+l}^{-(b_l - b_m + 1)} e^{\zeta(u) \frac{v\bar{v}'}{\rho^2}} \\
& \times \exp \left\{ \frac{1}{2} \zeta^2(u) \frac{(u\bar{v}' + v\bar{u})^2}{\rho^2} \frac{\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}}}{1 - \zeta(u) |u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})} \right\} \\
& \times \operatorname{erf} \left(\frac{\sqrt{\zeta(u)}}{\sqrt{1 - \zeta(u) |u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})}} \frac{u\bar{v}' + v\bar{u}}{\rho \sqrt{2} |u|} \right).
\end{aligned}$$

5.3. Scaling limits. Now, we are ready to prove the limiting eigenvalue density and local universality for products of truncated unitary matrices.

Notice that when $0 < |u| < 1$ for the solution $\xi(u)$ of the algebraic equation (5.16) and the solution $\zeta(u)$ of (5.30) we have the following relation

$$\zeta(u) = \frac{\tau_m}{\xi(u) - |u|^2}. \quad (5.52)$$

In this case

$$s_k = \sqrt{\tilde{s}_{m+k}/\tilde{s}_k}, \quad k = 1, \dots, m-1, \quad (5.53)$$

where s_k and \tilde{s}_k are defined respectively by (5.17) and (5.31).

In comparison with the limiting density in [3, eq.(3.12)] where all σ_k, b_k are zero and all τ_k are equal, we have a more general result.

Theorem 5.6. *For the weight function $w_m(z)$ given by (5.5), suppose that*

$$\sigma_k N + b_k > -1, \quad \sigma_k \geq 0 \quad \text{and} \quad \tau_k > 0 \quad \text{for } k = 1, \dots, m.$$

For $z \neq 0$, let $\zeta_m(z)$ be the largest real root of algebraic equation in x

$$|z|^2 \prod_{k=1}^m (\sigma_k - \sigma_m + \tau_k + x |z|^2) - \prod_{k=1}^m (\sigma_k - \sigma_m + x |z|^2) = 0. \quad (5.54)$$

Then the limiting eigenvalue density

$$\begin{aligned}
R_1(z) &:= \lim_{N \rightarrow \infty} \frac{1}{N} R_{N,1}(z) \\
&= \frac{1}{\pi |z|^2} \sum_{k=1}^m \left(\frac{1}{\sigma_k - \sigma_m + |z|^2 \zeta(z)} - \frac{1}{\sigma_k - \sigma_m + \tau_k + |z|^2 \zeta(z)} \right) \\
&\quad \times \chi \left\{ u: \sqrt{\frac{\sigma_1 \cdots \sigma_m}{(\sigma_1 + \tau_1) \cdots (\sigma_m + \tau_m)}} < |u| < \sqrt{\frac{(1+\sigma_1) \cdots (1+\sigma_m)}{(1+\sigma_1 + \tau_1) \cdots (1+\sigma_m + \tau_m)}} \right\} (z)
\end{aligned} \quad (5.55)$$

holds true for any complex number z such that $0 < |z| < 1$.

Proof. Taking $v = 0$ in Lemma 5.2, we have

$$\begin{aligned} w_m(z) &\sim \left(\frac{2\pi}{N}\right)^{(m-1)/2} \prod_{k=1}^m \frac{\Gamma((\sigma_k + \tau_k)N + b_k + 2)}{\Gamma(\sigma_k N + b_k + 1)\Gamma(\tau_k N + 1)} |z|^{2(\sigma_m N + b_m)} \\ &\times \frac{1}{\pi \sqrt{\left(1 + \frac{\tau_m |u|^2 \xi(u)}{(\xi(u) - |u|^2)^2} \sum_{k=1}^{m-1} \frac{(1-s_k^2)^2}{\tau_k s_k^2}\right) \prod_{k=1}^{m-1} \frac{\tau_k}{(1-s_k^2)^2}}} \prod_{k=1}^{m-1} s_k^{2(b_k - b_m) - 1} \\ &\times \exp\left\{N \sum_{k=1}^{m-1} (2(\sigma_k - \sigma_m) \ln s_k + \tau_k \ln(1 - s_k^2)) + N \tau_m \ln\left(1 - \frac{|u|^2}{\xi(u)}\right)\right\}. \end{aligned} \quad (5.56)$$

Taking $v, v' = 0$ in Lemmas 5.3, 5.4 and 5.5, we get the large N asymptotics of (5.33)

$$\begin{aligned} T_N(z, z) &\sim N^{2m+N} \sum_{k=1}^m \tau_k |z|^{-2(\sigma_m N + b_m)} (2\pi)^{-m} \prod_{k=1}^m \frac{\Gamma(\sigma_k N + b_k + 1)}{\Gamma(\sigma_k N + \tau_k N + b_k + 2)} \\ &\times \frac{\pi(2\pi/N)^{(2m-1)/2}}{\sqrt{(1 - \zeta(u) |u|^2 (\sum_{k=1}^m \frac{1}{\tilde{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\tilde{s}_{m+l}})) \prod_{k=1}^{2m-1} \frac{1}{\tilde{s}_k}}} \\ &\times \exp\left\{-N \sum_{k=1}^m (\tilde{s}_k - (\sigma_k - \sigma_m + \tau_k) \ln \tilde{s}_k) + N \zeta(u) |u|^2\right\} \\ &\times \exp\left\{N \sum_{k=1}^{m-1} (\tilde{s}_{m+k} - (\sigma_k - \sigma_m) \ln \tilde{s}_{m+k})\right\} \prod_{k=1}^m \tilde{s}_k^{b_k - b_m + 1} \prod_{l=1}^{m-1} \tilde{s}_{m+l}^{-(b_l - b_m + 1)} \\ &\times \chi\left\{y: \sqrt{\frac{\sigma_1 \cdots \sigma_m}{(\sigma_1 + \tau_1) \cdots (\sigma_m + \tau_m)}} < |y| < \sqrt{\frac{(1 + \sigma_1) \cdots (1 + \sigma_m)}{(1 + \sigma_1 + \tau_1) \cdots (1 + \sigma_m + \tau_m)}}\right\}(u). \end{aligned} \quad (5.57)$$

Noting $z = u$, since $R_{N,1}(z) = w_m(z)T_N(z, z)$, combination of (5.56) and (5.57) completes the proof. \square

Theorem 5.7. *With the same notation as in Theorem 5.6, let*

$$\rho = \frac{1}{|z|} \frac{1}{\sqrt{\sum_{k=1}^m \left(\frac{1}{\sigma_k - \sigma_m + |u|^2 \zeta(u)} - \frac{1}{\sigma_k - \sigma_m + \tau_k + |u|^2 \zeta(u)}\right)}} \quad (5.58)$$

and introduce rescaling variables

$$z_k = u + \frac{v_k}{\rho \sqrt{N}}, \quad k = 1, \dots, n. \quad (5.59)$$

Then the following hold true uniformly for v_1, \dots, v_n in any compact subset of \mathbb{C} .

(1) **Bulk limit.** For $\sqrt{\frac{\sigma_1 \cdots \sigma_m}{(\sigma_1 + \tau_1) \cdots (\sigma_m + \tau_m)}} < |u| < \sqrt{\frac{(1 + \sigma_1) \cdots (1 + \sigma_m)}{(1 + \sigma_1 + \tau_1) \cdots (1 + \sigma_m + \tau_m)}}$,

$$\begin{aligned} \lim_{N \rightarrow \infty} N^{-n} \rho^{-2n} R_{N,n}(z_1, \dots, z_n) \\ = \det \left(\frac{1}{\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (5.60)$$

(2) **Inner edge.** For $\delta_1, \dots, \delta_m > 0$ and $u = \sqrt{\frac{\sigma_1 \dots \sigma_m}{(\sigma_1 + \tau_1) \dots (\sigma_m + \tau_m)}} e^{i\phi}$ with $0 \leq \phi < 2\pi$,

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{-n} \rho^{-2n} R_{N,n}(z_1, \dots, z_n) \\ &= \det \left(\frac{1}{2\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \operatorname{erfc} \left(-\frac{e^{i\phi} \bar{v}_j + v_k e^{-i\phi}}{\sqrt{2}} \right) \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (5.61)$$

(3) **Outer edge.** For $u = \sqrt{\frac{(1+\sigma_1) \dots (1+\sigma_m)}{(1+\sigma_1 + \tau_1) \dots (1+\sigma_m + \tau_m)}} e^{i\phi}$ with $0 \leq \phi < 2\pi$,

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{-n} \rho^{-2n} R_{N,n}(z_1, \dots, z_n) \\ &= \det \left(\frac{1}{2\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \operatorname{erfc} \left(\frac{e^{i\phi} \bar{v}_j + v_k e^{-i\phi}}{\sqrt{2}} \right) \right)_{1 \leq k, j \leq n}. \end{aligned} \quad (5.62)$$

Proof. Let

$$\begin{aligned} \psi_N(v) &= \exp \left\{ \frac{1}{4} \zeta^2(u) \frac{(v\bar{u})^2 - (u\bar{v})^2}{\rho^2} \frac{\sum_{k=1}^m \frac{1}{\bar{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\bar{s}_{m+l}}}{1 - \zeta(u) |u|^2 \left(\sum_{k=1}^m \frac{1}{\bar{s}_k} - \sum_{l=1}^{m-1} \frac{1}{\bar{s}_{m+l}} \right)} \right\} \\ &\times \exp \left\{ \frac{1}{2} \sqrt{N} \zeta(u) \frac{v\bar{u} - u\bar{v}}{\rho} + \sigma_m \sqrt{N} \frac{u\bar{v} - v\bar{u}}{2|u|^2 \rho} - \sigma_m \frac{(u\bar{v})^2 - (v\bar{u})^2}{4|u|^4 \rho^2} \right\} \end{aligned} \quad (5.63)$$

and let $D = \operatorname{diag}(\psi_N(v_1), \dots, \psi_N(v_n))$.

In the bulk, combining Lemma 5.2, Lemma 5.3, Statement (I) in Lemma 5.4 and (I) in Lemma 5.5, we obtain

$$\psi_N(v_k) K_N(z_k, z_j) \psi_N^{-1}(v_j) \sim N \frac{1}{\pi} \rho^2 e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)}.$$

Furthermore,

$$\begin{aligned} R_{N,n}(z_1, \dots, z_n) &= \det (D(K_N(z_k, z_j)) D^{-1}) \\ &\sim N^n \rho^{2n} \det \left(\frac{1}{\pi} e^{-\frac{1}{2}(|v_k|^2 + |v_j|^2 - 2v_k \bar{v}_j)} \right)_{1 \leq k, j \leq n}. \end{aligned}$$

This completes the bulk limit of the theorem.

Likewise, combining Lemmas 5.2, 5.3, 5.4 and 5.5 we can prove the inner and outer edge cases. \square

Remark 5.8. Although we assume that all $\tau_k > 0$ in this section, our conclusions remain valid under the condition that at least one of the τ_k 's is positive. Indeed, in the latter case the asymptotics of truncated series remains true and asymptotics of weight functions almost corresponds to the case of those $\tau_k > 0$.

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APPENDIX A. DETERMINANT AND INVERSE OF ONE MATRIX

When calculating the Hessian matrix and the inverse we frequently encounter one type of matrix like

$$A := A(a_1, \dots, a_n) = \begin{pmatrix} 1 + a_1 & 1 & \cdots & 1 \\ 1 & 1 + a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 + a_n \end{pmatrix}. \quad (\text{A.1})$$

The determinant reads off

$$\det(A) = a_1 a_2 \cdots a_n \left(1 + \sum_{k=1}^n \frac{1}{a_k} \right). \quad (\text{A.2})$$

Moreover, if $a_1 a_2 \cdots a_n \neq 0$ and also $1 + \sum_{k=1}^n \frac{1}{a_k} \neq 0$, then the inverse A^{-1} is equal to

$$\frac{1}{1 + \sum_{k=1}^n \frac{1}{a_k}} \begin{pmatrix} \frac{1}{a_1} + \sum_{k \neq 1} \frac{1}{a_1 a_k} & -\frac{1}{a_1 a_2} & \cdots & -\frac{1}{a_1 a_n} \\ -\frac{1}{a_2 a_1} & \frac{1}{a_2} + \sum_{k \neq 2} \frac{1}{a_2 a_k} & \cdots & -\frac{1}{a_2 a_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{a_n a_1} & -\frac{1}{a_n a_2} & \cdots & \frac{1}{a_n} + \sum_{k \neq n} \frac{1}{a_n a_k} \end{pmatrix}. \quad (\text{A.3})$$

Equivalently,

$$A^{-1} = -\frac{1}{1+s} \text{diag}\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right) A (-a_1 + a_1 s, \dots, -a_n + a_n s) \text{diag}\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right) \quad (\text{A.4})$$

where

$$s = \sum_{k=1}^n \frac{1}{a_k}. \quad (\text{A.5})$$

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